Gravitational Radiation in the Brans-Dicke and Rosen bi-metric Theories of Gravity with a Comparison with General Relativity[1]

by

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ABSTRACT

General relationships are developed for gravitational radiation in the weak field approximation in the Brans-Dicke scalar-tensor theory and the Rosen bi-metric theory. Both periodic and aperiodic systems are considered, with results for the former being of quadrupole order. The specific cases of a binary orbital system and a system of colliding particles are treated. An attempt to test the validity of the Brans-Dicke, Rosen, and general relativistic theories is made by applying results to the observed binary pulsar PSR 1913+16.

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CONVENTIONS

,	partial derivative
;	covariant derivative w.r.t. $g_{\mu\nu}$
|	covariant derivative w.r.t. $\gamma_{\mu\nu}$
$\eta_{\mu\nu}$	diag. (-1,1,1,1) = Minkowski metric.
g	$-\ \det.\left(g_{\mu\nu}\right)$
γ	$-\ \det.\left(\gamma_{\mu\nu}\right)$
$\Box^2\psi$	$\psi_{;\lambda;}{}^{\lambda}$
$R_{\mu\nu}$	Ricci tensor
$T_{\mu\nu}$	energy-momentum tensor of matter
$t_{\mu\nu}$	gravitational energy-momentum pseudo tensor
$\mu,\nu,\lambda\ \ldots$	index values 0, 1, 2, 3
$i, j, k\ \ldots$	index values 1, 2, 3
$d\tau^2$	$-g_{\mu\nu}dx^{\mu}dx^{\nu}$
$\langle\ \rangle$	time average
$\vec{a}$	3-vector $\left(a_1, a_2, a_3\right)$

1.0 **Introduction**

In recent months there has been an upsurge of interest in the theoretical prediction of gravitational radiation in general relativity following observations by Taylor *et al.* (1) on the binary pulsar PSR 1913+16.[3, 4] They have found a systematic decrease of the orbital period of the system that is consistent with energy loss due to gravitational radiation as predicted by Einstein's general theory of relativity. The compact nature of the participating objects is such as to rule out convincingly significant contributions from other mechanisms such as tidal interaction. These observations represent the first tests of general relativity outside the solar system and also constitute the first convincing experimental evidence, though indirect, for gravitational waves.

These observations simultaneously raise the question as to whether it might not also be possible to discriminate, on the basis of the same data, between general relativity and other competing theories of gravity whose predictions within the solar system are indistinguishable from Einstein's theory. Here we consider gravitational radiation in the two theories that currently represent viable alternatives: the Brans-Dicke scalar-tensor theory (2,3) and the Rosen bi-metric theory (4,5).

There are principally two distinct methods whereby gravitational radiation can be estimated in the theories. First, the EIH (Einstein, Infeld, Hoffmann) method (6) consists in solving the equations of motion in a power series in a suitable parameter such as v/c . The method is recursive and in principle converges to the exact solution. Gravitational radiation can be estimated by using the motion so derived to deduce the rate of change of total energy and by assuming that any decrease that is not accounted for by other means goes off as gravitational radiation. The second method, the weak field approximation (7), consists in linearizing the field equations by approximating to the case in which gravitational effects are everywhere small. The field equations then reduce to linear wave equations from which radiation can be deduced directly. Results are not in principle exact, being only as good as the validity of the weak field approximation itself.

In the EIH method, radiation is inferred to the order of recursion to which the theorist is willing or capable of going, with exact results possible in principle in the limit. In the weak field approximation, radiation is seen directly and in many cases can be computed exactly within the linearized theory, but may or may not be exact in terms of the correct

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[3] This paper is written from the perspective of 1979.

[4] Joseph H. Taylor, Jr., and Russell A. Hulse received the Nobel Prize in Physics in 1993 "for the discovery of a new type of pulsar, a discovery that has opened up new possibilities for the study of gravitation".

nonlinear theory. So far it has not been possible to find a satisfactory theoretical bridge between the predictions of the two methods, and the problem remains open. This situation has resulted in an ongoing controversy as to the correct radiation rate predicted by general relativity (8). It is possible that predictions made in the competing theories may establish bounds on the radiation rate which will be useful in clarifying the relationship between the EIH and weak field methods.

The general theory of relativity predicts radiation that, in the lowest order, is proportional to the third derivative of the quadrupole moment of the mass-energy distribution. This prediction follows in either of the two methods. It is a consequence of the conservation equations that the first derivative of the monopole moment and the second derivative of the dipole moment are zero, so that radiation is first seen in the quadrupole term. In the alternative theories the situation is different.

The essential feature of the Brans-Dicke theory is that the gravitational "constant" G is in fact not a constant but is determined by the totality of matter in the universe through an auxiliary scalar field equation. G expresses the ability of mass-energy to interact gravitationally. The non-universality of G means that the effective interaction strength of a quantity of mass-energy is determined by the local value of the scalar G field. To make an analogy with electromagnetic theory, the effective gravitational "charge-to-mass" ratio is not a constant. For the same reason that the existence of differing charge-to-mass ratios among the particles produces non-vanishing electric dipole moments, the variation of G in the Brans-Dicke theory introduces dipole terms in the EIH gravitational radiation equations.

Will (9) has computed the radiation due to these terms in the Brans-Dicke theory for both the scalar and the tensor field. Eardley (l0) has investigated the effect of these terms on PSR 1913+16. The extent of the dipole effect depends on the difference of the self-gravitational binding energy per unit mass for the two bodies and is thus dependent also on the internal structure of the objects. When the objects are in circular orbits, the time variation of the scalar field at each object due to the motion of the other is zero and the dipole contributions consequently drop out. Under these circumstances the dominant surviving terms are of quadrupole order.

Here we develop expressions for quadrupole gravitational radiation in the Brans-Dicke theory using the weak field technique and apply these results, which are applicable in general, to the specific example of PSR 1913+16, though its orbit is eccentric. We also use the weak field approach to compute the power spectral density to all multipole orders associated with a system of colliding particles. This result is useful as a basis for the study of gravitational radiation emitted by a hot gas in the Brans-Dicke theory.

Dipole radiation also appears in the application of the EIH method to the bi-metric theory of Rosen. In Rosen's theory there are two metrics: one that describes purely gravitational effects, as is general relativity, and a second that accounts for inertial effects independent of gravity. While the gravitational constant does not vary with space-time position as in the Brans-Dicke theory, the energy-momentum tensor in the field equations is scaled by the square-root of the ratio of the determinants of the two metrics. Thus the effective mass-energy is dependent on the local field, and dipole radiation follows as a consequence as in the Brans-Dicke theory. Again the dipole radiation is a function of the internal structure of the participating bodies and is zero in certain circumstances such as the case of circular orbits. For this reason, it is again of some interest to compute the radiation rate due to the quadrupole term.

Will and Eardley (12) have computed the dipole radiation rate in the bi-metric theory and have found that it carries negative energy. That is, the dipole term acts to increase, rather than decrease, the energy of the system. Rosen (13) has argued that it is possible to assume a time-symmetric solution, rather than the retarded solution used by Will and Eardley, with the consequence that there is predicted no energy gain or loss due to radiation of any order.

We investigate here quadrupole gravitational radiation in Rosen's bi-metric theory using the conventional retarded solution in the weak field approximation. As in the Brans-Dicke theory, we also investigate the power spectral density associated with a system of colliding particles.

2.0 **Quadrupole Gravitational Radiation in the Brans-Dicke Theory**

General Results

Field Equations

The field equations in the Brans-Dicke theory (2,3) are

$$\Box^2 \varphi = \frac{8\pi}{3+2\omega} T^{\lambda}{}_{\lambda} \tag{2.1}$$

and

$$R_{\mu\nu} = -\frac{8\pi}{\varphi}\left[T_{\mu\nu} - g_{\mu\nu}\frac{1+\omega}{3+2\omega}T^{\lambda}{}_{\lambda}\right] - \frac{1}{\varphi}\varphi_{;\mu;\nu} - \frac{\omega}{\varphi^2}\varphi_{;\mu}\varphi_{;\nu} \tag{2.2}$$

The Ricci tensor $R_{\mu\nu}$ is formed from the metric tensor $g_{\mu\nu}$ as in general relativity. φ is the scalar field, determined by the auxiliary equation (2.1), which plays the role of G . ω is a free parameter of the theory whose value cannot be determined *a priori*. In fact, one of the objectives of developing here the relationships for quadrupole radiation is to determine if observations such as those of Taylor *et al.* on PSR 1913+16 can be used to establish bounds on ω .

In addition to the field equations (2.1) and (2.2), we have the conservation law in the Brans-Dicke theory

$$T^{\mu\nu}{}_{;\nu} = 0 \tag{2.3}$$

as in general relativity.

Finally, it can be shown that the energy-momentum tensor associated with the scalar field φ is

$$T_{(\varphi)\mu\nu} = \frac{\omega}{8\pi\varphi}\left(\varphi_{;\mu}\varphi_{;\nu} - \frac{1}{2}g_{\mu\nu}\varphi_{;\rho}\varphi_{;}{}^{\rho}\right) + \frac{1}{8\pi}\left(\varphi_{;\mu;\nu} - g_{\mu\nu}\Box^2\varphi\right) \tag{2.4}$$

The Brans-Dicke theory is motivated by an attempt to incorporate Mach's principle into relativity. The view that we adopt here is that the system whose radiation we wish to compute is embedded within the rest of the mass-energy of the universe which, for our purposes, is static. Therefore φ , which is related to G , consists of a dominant constant term φ_0 determined by the rest of the universe and a perturbative term ψ induced by our system.

$$\varphi = \varphi_0 + \psi \tag{2.5}$$

$$\varphi_0 = \text{constant}, \qquad \psi = \text{"small"}.$$

<u>Weak Field Limit - Coordinate Condition</u>

The passage to the weak field limit is achieved by assuming that the space-time metric $g_{\mu\nu}$ is everywhere adequately represented by

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{2.6}$$

where $h_{\mu\nu}$ is "small" and $\eta_{\mu\nu}$ is the Minkowski metric of flat spacetime. By definition,

$$g_{\mu\nu} g^{\nu\lambda} = \delta_{\mu}{}^{\lambda}. \tag{2.7}$$

This fact and (2.6) imply that

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}. \tag{2.8}$$

To the order with which we are concerned, all raising and lowering of indices is with $\eta_{\mu\nu}$ and covariant derivatives go over to ordinary partial derivatives.

Equations (2.1) and (2.5) give immediately,

$$\Box^2 \psi = \frac{8\pi}{3 + 2\omega} T^{\lambda}{}_{\lambda}. \tag{2.9}$$

The use of (2.6) and (2.8) in $R_{\mu\nu}$ leads, with (2.2), to the weak field tensor equation

$$2 R^{(1)}_{\mu\nu} = \Box^2 h_{\mu\nu} + h^{\lambda}_{\lambda,\mu,\nu} - h^{\lambda}_{\mu,\lambda,\nu} - h^{\lambda}_{\nu,\lambda,\mu} = -16\pi S_{\mu\nu} \tag{2.10}$$

where

$$S_{\mu\nu} \equiv \varphi_0^{-1}\left[T_{\mu\nu} - \eta_{\mu\nu}\frac{1+\omega}{3+2\omega}T^{\lambda}{}_{\lambda}\right] + \frac{1}{8\pi}\varphi_0^{-1}\psi_{,\mu,\nu} \tag{2.11}$$

to first order in ψ and $h_{\mu\nu}$. $R_{\mu\nu}^{(N)}$ denotes the term in $R_{\mu\nu}$ that is of N-th order in $h_{\mu\nu}$.

The solution of the system (2.9) and (2.10) would be considerably simplified if the scalar field ψ did not couple into the tensor field $h_{\mu\nu}$ by appearing in the source term $S_{\mu\nu}$. Because of the Bianchi identities

$$\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)_{;\nu} = 0$$

there are four degrees of freedom that are not uniquely specified by the tensor field equations. Consequently we have at our disposal four coordinate, or gauge, conditions that can be used to great advantage.

By inspection of (2.10) and (2.11), the term involving ψ could be eliminated if

$$h^{\lambda}_{\lambda,\mu,\nu} - h^{\lambda}_{\mu,\lambda,\nu} - h^{\lambda}_{\nu,\lambda,\mu} = -2\varphi_0^{-1}\psi_{,\mu,\nu}\,. \tag{2.12}$$

If we demand that the following coordinate condition be satisfied

$$h^{\lambda}_{\mu,\lambda} - \frac{1}{2}h^{\lambda}_{\lambda,\mu} = \varphi_0^{-1}\psi_{,\mu}\,, \tag{2.13}$$

then (2.12) will in fact be true. To see this, take the derivative of (2.13) with respect to x^{ν},

$$h^{\lambda}_{\mu,\lambda,\nu} - \frac{1}{2}h^{\lambda}_{\lambda,\mu,\nu} = \varphi_0^{-1}\psi_{,\mu,\nu}\,.$$

Now exchange μ and ν and utilize the commutivity of partial differentiation to get

$$h^{\lambda}_{\nu,\lambda,\mu} - \frac{1}{2}h^{\lambda}_{\lambda,\mu,\nu} = \varphi_0^{-1}\psi_{,\mu,\nu}\,.$$

The sum of these two expressions, with the overall sign reversed, is just (2.12).

The condition (2.13), incidentally, goes over to the weak field version of the so-called harmonic coordinate condition if the scalar field ψ goes to zero.

With (2.12) established by the coordinate condition (2.13), the tensor field equation (2.10) is now considerably simplified.

$$\Box^2 h_{\mu\nu} = -16\pi\varphi_0^{-1}\left[T_{\mu\nu} - \eta_{\mu\nu}\frac{1+\omega}{3+2\omega}T^{\lambda}{}_{\lambda}\right] \tag{2.14}$$

Of course, the scalar and tensor fields, though now apparently uncoupled, are in fact still coupled (as they must be) through the imposition of the coordinate condition (2.13). Our objective now is to solve the field equations (2.9) and (2.14) and to use the results to evaluate the energy momentum tensors associated with the two fields. But before doing so we digress to establish the relationship between φ_0 and G .

<u>Relationship between φ_0 and G.</u>

Consider a very simple static system with zero pressure. Let

$$T_{\mu\nu} = \mathrm{diag}\left(\rho,0,0,0\right) \Rightarrow T^{\lambda}{}_{\lambda} = -\rho \ .$$

The tensor field equation (2.14) with $\partial_0 = \partial_t = 0$ becomes

$$\Box^2 h_{00} = \nabla^2 h_{00} = -16\pi\varphi_0^{-1}\left[\rho - (-1)\frac{1+\omega}{3+2\omega}\left(-\rho\right)\right] = -16\pi\varphi_0^{-1}\rho\left(\frac{2+\omega}{3+2\omega}\right).$$

For a general, but localized, distribution $\rho(\vec{x})$, h_{00} is

$$h_{00} = 4\varphi_0^{-1}\left(\frac{2+\omega}{3+2\omega}\right)\int d^3\vec{x}'\frac{\rho(\vec{x}')}{|\vec{x}'-\vec{x}|} \cong 4\varphi_0^{-1}\left(\frac{2+\omega}{3+2\omega}\right)\frac{M}{r}$$

where $\left|\vec{x}'-\vec{x}\right| \cong r$ and

$$M \equiv \int d^3\vec{x}'\rho(\vec{x}') \tag{2.15}$$

We conclude from (2.6) that, in this case,

$$g_{00} = \eta_{00} + h_{00} = -1 + \frac{2M}{r}\varphi_0^{-1}\frac{4+2\omega}{3+2\omega}.$$

If this is to go over to the Newtonian result for large *r*,

$$g_{00} \to -1 - 2\Phi = -1 + \frac{2MG}{r},$$

it must be that φ_0 and G are related by

$$\varphi_0 = \frac{1}{G}\left(\frac{4+2\omega}{3+2\omega}\right). \qquad (2.16)$$

When $\omega \to \infty$, $\varphi_0^{-1} \to G$, and the Brans-Dicke theory goes over to general relativity.

Energy-Momentum of the Gravitational Field - Tensor Part

Let us assume that $T_{\mu\nu}$ is localized to a finite region. Outside that region $T_{\mu\nu} = 0$. Then, as a consequence of (2.10) and (2.12),

$$R^{(1)}_{\mu\nu} = \Box^2 h_{\mu\nu} = 0 \qquad (2.17)$$

outside the region. There are many ways to define the energy-momentum tensor of the gravitational field. The one most applicable to the present weak field situation is to consider $R_{\mu\nu}$ on the left side of (2.2) to consist of a series of terms in $R^{(N)}_{\mu\nu}$. In the development of the weak field equation (2.14), we have explicitly left $R^{(1)}_{\mu\nu}$ on the left side. The remaining higher order terms, which so far have been ignored, could be brought to the right side. If the source region gives rise to a flux of energy in the form of gravitational waves, it must be represented in these higher order terms. Since, outside the source, (2.17) is true, the lowest order non-zero additional term appearing now on the right side of (2.14) that could possibly represent such a flux turns out to be

$$t_{\mu\nu} = \frac{\varphi_0}{8\pi}\left[R^{(2)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\lambda\rho}R^{(2)}_{\lambda\rho}\right]. \qquad (2.18)$$

We take this as our definition of the gravitational energy-momentum tensor associated with the tensor field $h_{\mu\nu}$. $R^{(2)}_{\mu\nu}$ is found to be

$$R^{(2)}_{\mu\nu} = -\frac{1}{2} h^{\lambda\kappa}\left[h_{\lambda\kappa,\nu,\mu} - h_{\mu\kappa,\nu,\lambda} - h_{\lambda\nu,\kappa,\mu} + h_{\mu\nu,\kappa,\lambda}\right] +$$

$$+\frac{1}{4}\left[2\, h^{\kappa}_{\sigma,\kappa} - h^{\kappa}_{\kappa,\sigma}\right]\left[h^{\sigma}_{\mu,\nu} + h^{\sigma}_{\nu,\mu} - h_{\mu\nu,}{}^{\sigma}\right] -$$

$$-\frac{1}{4}\left[h_{\sigma\nu,\lambda} + h_{\sigma\lambda,\nu} - h_{\lambda\nu,\sigma}\right]\left[h^{\sigma}{}_{\mu,}{}^{\lambda} + h^{\sigma\lambda}{}_{,\mu} - h^{\lambda}{}_{\mu,}{}^{\sigma}\right] \qquad (2.19)$$

Far Field Approximation

The expression for the gravitational tensor $t_{\mu\nu}$ defined by (2.18) and (2.19) can be simplified considerably by taking advantage of approximations valid very far from the source region.

We anticipate that very far from the source both ψ and $h_{\mu\nu}$ will be, approximately, functions of a single scalar variable t'

$$t' = t - r \qquad (2.20)$$

where

$$r^2 = x_i x^i . \qquad (2.21)$$

Such a scalar can be constructed from the vector x^μ by forming

$$t' = k_\lambda x^\lambda \qquad (2.22)$$

with

$$k_0 = -k^0 \equiv +1 , \qquad k_i \equiv -\hat{x}_i \qquad (2.23)$$

$$\hat{x}^i \equiv \frac{x^i}{r} \qquad (2.24)$$

Over a suitably small region in the far field k_λ can be regarded as a constant vector. That is, ψ and $h_{\mu\nu}$ will be approximately plane. Any $1/r$ variation or variation of the unit vector $\hat{x}$ over points in the region can be made arbitrarily small by taking the region to be sufficiently far from the source. The dominant functional dependency of our solutions will be on the scalar $t' = t - r$. This fact can be exhibited explicitly by expressing all partials of ψ and $h_{\mu\nu}$ as

$$h_{\mu\nu,\sigma} = \frac{\partial t'}{\partial x^{\sigma}} \frac{dh_{\mu\nu}}{dt'} = k_{\lambda}\delta_{\sigma}^{\lambda}\dot{h}_{\mu\nu} = k_{\sigma}\dot{h}_{\mu\nu} \tag{2.25}$$

and

$$\psi_{,\sigma} = k_{\sigma}\dot{\psi} \tag{2.26}$$

where

$$h_{\mu\nu} = h_{\mu\nu}\left(k_{\lambda}x^{\lambda}\right) = h_{\mu\nu}(t'), \tag{2.27}$$

$$\psi = \psi\left(k_{\lambda}x^{\lambda}\right) = \psi(t'), \tag{2.28}$$

$$\dot{f}(t) \equiv \frac{df(t)}{dt}, \tag{2.29}$$

$$\frac{\partial x^{\lambda}}{\partial x^{\sigma}} = \delta_{\sigma}^{\lambda}. \tag{2.30}$$

Again, because $T_{\mu\nu} = 0$ outside the source region,

$$\Box^2 h_{\mu\nu} = 0 \qquad \text{and} \qquad \Box^2 \psi = 0 \tag{2.31}$$

in the far field. If (2.25) is used in the first of these, we find

$$\Box^2 h_{\mu\nu} = h_{\mu\nu,\rho,}{}^{\rho} = \left(k_{\rho}\dot{h}_{\mu\nu}\right)_{,}{}^{\rho} = k_{\rho}k^{\rho}\ddot{h}_{\mu\nu},$$

implying that

$$k_{\rho}k^{\rho} = 0. \tag{2.32}$$

Obviously the same condition follows from the second of (2.31).

We may now use (2.25) and (2.26) to express the coordinate condition (2.13) in the far field. From (2.13) we have

$$k_\lambda \dot{h}^\lambda{}_\mu = k_\mu\left(\varphi_0^{-1}\dot{\psi} + \frac{1}{2}\dot{h}^\lambda{}_\lambda\right). \tag{2.33}$$

Since any constant of integration is of no consequence in the radiation of energy, (2.33) can be integrated or differentiated by inspection. In particular

$$k_\lambda h^\lambda{}_\mu = k_\mu\left(\varphi_0^{-1}\psi + \frac{1}{2}h^\lambda{}_\lambda\right) \tag{2.34}$$

and

$$k_\lambda \ddot{h}^\lambda{}_\mu = k_\mu\left(\varphi_0^{-1}\ddot{\psi} + \frac{1}{2}\ddot{h}^\lambda{}_\lambda\right). \tag{2.35}$$

These relations stemming from the coordinate condition can be used to effect a simplification of $R^{(2)}_{\mu\nu}$ (2.19). First, we notice that the first factor of the second term of (2.19) is just the coordinate condition itself,

$$2h^\kappa_{\sigma,\kappa} - h^\kappa_{\kappa,\sigma} = 2\varphi_0^{-1}k_\sigma\dot{\psi}\ .$$

For the rest of (2.19), the objective is to use (2.33), (2.34) and (2.35) to draw out of each term explicit dependency on $k_\mu k_\nu$. For example,

$$h^{\lambda\kappa}h_{\mu\nu,\kappa,\lambda} = \underset{\uparrow}{h}{}^{\lambda\kappa}k_\kappa \underset{\uparrow}{k_\lambda}\ddot{h}_{\mu\nu} = k^\kappa k_\kappa\left(\varphi_0^{-1}\psi + \frac{1}{2}h^\lambda{}_\lambda\right)\ddot{h}_{\mu\nu} = 0$$

from (2.32), and

$$h_{\lambda\nu,\sigma}h^\sigma{}_{\mu,}{}^\lambda = \underset{\uparrow}{k_\sigma}\overset{\downarrow}{\dot{h}}_{\lambda\nu}\overset{\downarrow}{k}{}^\lambda \underset{\uparrow}{\dot{h}}{}^\sigma{}_\mu = k_\mu k_\nu\left(\varphi_0^{-1}\dot{\psi} + \frac{1}{2}\dot{h}^\lambda{}_\lambda\right)^2$$

To ease the notation let

$$z \equiv \varphi_0^{-1}\psi + \frac{1}{2}h^\lambda{}_\lambda\,. \tag{2.36}$$

Then (2.33), (2.34), and (2.35) imply that

$$R^{(2)}_{\mu\nu} = k_\mu k_\nu\left[z\ddot{z} - \frac{1}{2}h^{\lambda\kappa}\ddot{h}_{\lambda\kappa} + \varphi_0^{-1}\dot{\psi}\,\dot{z} - \frac{1}{4}\dot{h}_{\lambda\kappa}\dot{h}^{\lambda\kappa} + \frac{1}{2}\dot{z}^2\right] \tag{2.37}$$

in the far field.

$h^{\lambda}{}_{\lambda}$ and ψ can be related to each other in the following way. From the scalar field equation (2.9) we have

$$\psi(\vec{x},t) = -\frac{2}{3+2\omega}\int d^3\vec{x}'\,\frac{T^{\lambda}_{\lambda}\left(\vec{x}',t-|\vec{x}'-\vec{x}|\right)}{|\vec{x}'-\vec{x}|}, \tag{2.38}$$

and from the tensor field equation (2.14) we have

$$h_{\mu\nu}(\vec{x},t) = 4\varphi_0^{-1}\left[\int d^3\vec{x}'\,\frac{T_{\mu\nu}\left(\vec{x}',t-|\vec{x}'-\vec{x}|\right)}{|\vec{x}'-\vec{x}|} - \right.$$
$$\left. -\eta_{\mu\nu}\frac{1+\omega}{3+2\omega}\int d^3\vec{x}'\,\frac{T^{\lambda}_{\lambda}\left(\vec{x}',t-|\vec{x}'-\vec{x}|\right)}{|\vec{x}'-\vec{x}|}\right]. \tag{2.39}$$

Contraction of (2.39) gives

$$h^{\lambda}_{\lambda}(\vec{x},t) = -4\varphi_0^{-1}\frac{1+2\omega}{3+2\omega}\int d^3\vec{x}'\,\frac{T^{\lambda}_{\lambda}\left(\vec{x}',t-|\vec{x}'-\vec{x}|\right)}{|\vec{x}'-\vec{x}|}. \tag{2.40}$$

By comparing (2.38) and (2.40) we find immediately that

$$h^{\lambda}_{\lambda}(\vec{x},t) = 2\varphi_0^{-1}(1+2\omega)\psi(\vec{x},t), \tag{2.41}$$

and when this is used in the three terms of (2.37), which involve z, we find that

$$R^{(2)}_{\mu\nu} = k_\mu k_\nu\left[-\frac{1}{2}\left(h^{\lambda\kappa}\ddot{h}_{\lambda\kappa} + \frac{1}{2}\dot{h}^{\lambda\kappa}\dot{h}_{\lambda\kappa}\right) + 2(1+\omega)\varphi_0^{-2}\left[2(1+\omega)\psi\ddot{\psi} + (2+\omega)\dot{\psi}^2\right]\right]. \tag{2.42}$$

The second term of the definition (2.18) of $t_{\mu\nu}$ involves

$$\eta^{\lambda\rho}R^{(2)}_{\lambda\rho} = \eta^{\lambda\rho}k_\lambda k_\rho[\cdots] = k^\rho k_\rho[\cdots] = 0$$

by (2.32). Therefore, from (2.18) and (2.42), the energy-momentum tensor associated with the tensor part of the gravitational field is

$$t_{\mu\nu} = \frac{k_\mu k_\nu}{8\pi}\begin{Bmatrix} -\frac{1}{2}\varphi_0\left(h^{\lambda\kappa}\ddot{h}_{\lambda\kappa} + \frac{1}{2}\dot{h}^{\lambda\kappa}\dot{h}_{\lambda\kappa}\right) + \\ +2(1+\omega)\varphi_0^{-1}\left[2(1+\omega)\psi\ddot{\psi} + (2+\omega)\dot{\psi}^2\right] \end{Bmatrix} \tag{2.43}$$

Averaging of the Field

Thus far we have incorporated approximations valid for a weak field far from the source, but have not sacrificed temporal detail. We could, if we wished, compute the instantaneous dE/dt with (2.43) as a basis. However, such an attempt would be extraordinarily complex and would not be useful. The overall effect on a system, such as a binary, is better characterized by the average flux of energy away from the system.

Suppose that the $h_{\mu\nu}$ (or ψ) waves have a discrete spectra1 representation. It is clear then that the longest periodicity T that can be seen temporally will be proportional to the inverse of the difference of the closest pair of frequency components in the wave. Therefore, it would be appropriate to evaluate the average of dE/dt over an interval equal to or greater than T. It turns out that the loss of temporal detail introduced by such averaging will simplify our final results strikingly.

A large class of problems, of which the binary system is an example, have discrete spectral representations so that the averaging process just described is applicable. Other problems, such as the collision of particles to be considered later, are not periodic but are represented by continuous spectra. In these cases the "longest periodicity" becomes infinitely long and averaging no longer has a practical meaning. But it is possible to define the power spectral density associated with the wave in such cases and we will find that this too will result in a considerable simplification of final results.

The instantaneous flux of energy through a surface of area $r^2 d\Omega$ in the direction $\hat{x}$ is

$$\frac{dE}{dt} = r^2 d\Omega\, \hat{x}^i t^{0i} \tag{2.44}$$

and the average flux is

$$\left\langle \frac{dE}{dt} \right\rangle = r^2 d\Omega\, \hat{x}^i \left\langle t^{0i} \right\rangle . \tag{2.45}$$

Consider, in particular, the average of the first term of (2.43), which involves

$$\left\langle h^{\lambda\kappa}\ddot{h}_{\lambda\kappa}\right\rangle = \frac{1}{T}\int_0^T dt' h^{\lambda\kappa}(t')\frac{d^2 h_{\lambda\kappa}(t')}{dt'^2} =$$

$$= \frac{1}{T}h^{\lambda\kappa}(t')\frac{dh_{\lambda\kappa}(t')}{dt'}\bigg|_0^T - \frac{1}{T}\int_0^T dt'\frac{dh^{\lambda\kappa}}{dt'}\frac{dh_{\lambda\kappa}}{dt'} . \tag{2.46}$$

If $h^{\lambda\kappa}$ and its first derivative are reasonably behaved functions, the first term of the partial integration above can be made arbitrarily small by letting T be sufficiently large. For purposes of averaging then we can drop the first term and set

$$\left\langle h^{\lambda\kappa}\ddot{h}_{\lambda\kappa}\right\rangle = -\left\langle \dot{h}^{\lambda\kappa}\dot{h}_{\lambda\kappa}\right\rangle . \tag{2.47}$$

In exactly the same way we conclude that

$$\left\langle \psi\ddot{\psi}\right\rangle = -\left\langle \dot{\psi}^2\right\rangle . \tag{2.48}$$

These results in (2.43) give

$$\left\langle t_{\mu\nu}\right\rangle = \frac{k_\mu k_\nu}{8\pi}\left\langle \frac{1}{4}\varphi_0\dot{h}^{\lambda\kappa}\dot{h}_{\lambda\kappa} - 2\omega(1+\omega)\varphi_0^{-1}\dot{\psi}^2\right\rangle . \tag{2.49}$$

Finally, it will be useful to re-write $\left\langle t_{\mu\nu}\right\rangle$ in terms of a function $J_{\mu\nu}$ defined to be

$$J_{\mu\nu}(\vec{x},t) \equiv 4\varphi_0^{-1}\int d^3\vec{x}'\,\frac{T_{\mu\nu}(\vec{x}',t-|\vec{x}'-\vec{x}|)}{|\vec{x}'-\vec{x}|} . \tag{2.50}$$

When this is used for the first term of (2.39), and (2.38) is used in the second, there results

$$h_{\mu\nu}(\vec{x},t) = J_{\mu\nu}(\vec{x},t) + 2\varphi_0^{-1}\eta_{\mu\nu}(1+\omega)\psi(\vec{x},t), \tag{2.51}$$

and consequently

$$\dot{h}^{\lambda\kappa}\dot{h}_{\lambda\kappa} = \dot{J}^{\lambda\kappa}\dot{J}_{\lambda\kappa} + 4\varphi_0^{-1}(1+\omega)\dot{\psi}\eta^{\lambda\kappa}\dot{J}_{\lambda\kappa} + 16\varphi_0^{-2}(1+\omega)^2\dot{\psi}^2 . \tag{2.52}$$

By contracting (2.50), and comparing with (2.38), we find

$$\eta^{\lambda\kappa} J_{\lambda\kappa} = J^{\lambda}{}_{\lambda} = -2\varphi_0^{-1}(3+2\omega)\psi \ , \tag{2.53}$$

and consequently that

$$\eta^{\lambda\kappa} \dot{J}_{\lambda\kappa} = \dot{J}^{\lambda}{}_{\lambda} = -2\varphi_0^{-1}(3+2\omega)\dot{\psi} \ . \tag{2.54}$$

This last result may be used to eliminate $\dot{\psi}$ from (2.52) and (2.49) to give

$$\left\langle t_{\mu\nu} \right\rangle = \frac{k_\mu k_\nu}{8\pi G} \frac{2+\omega}{3+2\omega} \left\langle \frac{1}{2} \dot{J}^{\lambda\kappa} \dot{J}_{\lambda\kappa} - \left(\frac{1+\omega}{3+2\omega} \right)^2 \left(\dot{J}^{\lambda}{}_{\lambda} \right)^2 \right\rangle, \tag{2.55}$$

where we have used also (2.16) for φ_0 .

Conservation Law and Moments

Analysis of the radiation in terms of multipoles is a result of one further approximation: the expansion of $J_{\mu\nu}$ in a Taylor series about $t' = t - r$. That is,

$$J^{\mu\nu}(\vec{x},t) = 4\varphi_0^{-1} \frac{1}{r} \left[\int d^3\vec{x}'\, T^{\mu\nu}(\vec{x}',t') + \hat{x} \cdot \int d^3\vec{x}'\, \vec{x}' \frac{\partial T^{\mu\nu}(\vec{x}',t')}{\partial t'} + \right.$$

$$\left. + \frac{1}{2} \int d^3\vec{x}' (\hat{x}\cdot\vec{x}')^2 \frac{\partial^2 T^{\mu\nu}(\vec{x}',t')}{\partial t'^2} + \cdots \right], \tag{2.56}$$

where we have used

$$\left|\vec{x}' - \vec{x}\right|^{-1} \cong 1/r \qquad \text{and} \qquad \left|\vec{x}' - \vec{x}\right| \cong r - \hat{x}\cdot\vec{x}'$$

for $r >> |\vec{x}'|$.

Let us define the following moments of the mass-energy distribution:

$$M(t) \equiv \int d^3\vec{x}\, T^{00}(\vec{x},t), \tag{2.57}$$

$$D^k(t) \equiv \int d^3\vec{x}\, x^k T^{00}(\vec{x},t), \tag{2.58}$$

$$Q^{ij}(t) \equiv \int d^3\vec{x}\, x^i x^j T^{00}(\vec{x},t). \tag{2.59}$$

The conservation law (2.3) becomes, in the weak field limit,

$$T^{\mu\nu}{}_{,\nu} = 0 \tag{2.60}$$

and implies the relations [See Appendix A]

$$\int d^3\vec{x}\, T^{jk}(\vec{x},t) = \frac{1}{2}\frac{\partial^2}{\partial t^2}\int d^3\vec{x}\, x^j x^k T^{00}(\vec{x},t) = \frac{1}{2}\ddot{Q}^{jk}(t) \tag{2.61}$$

$$\int d^3\vec{x}\, T^{0k}(\vec{x},t) = \frac{\partial}{\partial t}\int d^3\vec{x}\, x^k T^{00}(\vec{x},t) = \dot{D}^k(t) \tag{2.62}$$

$$\frac{\partial}{\partial t}\int d^3\vec{x}\, x^k T^{j0}(\vec{x},t) = \int d^3\vec{x}\, T^{jk}(\vec{x},t) = \frac{1}{2}\ddot{Q}^{jk}(t). \tag{2.63}$$

We use the expansion (2.56) of $J^{\mu\nu}$ to write J^{00}, J^{0i}, and J^{ij} in terms of the moments (2.57) - (2.59).

First, from (2.56) we have

$$J^{00}(\vec{x},t) = 4\varphi_0^{-1}\frac{1}{r}\left[\int d^3\vec{x}'\, T^{00}(\vec{x}',t') + \hat{x}_i \frac{\partial}{\partial t'}\int d^3\vec{x}'\, x'^i T^{00}(\vec{x}',t') + \right.$$

$$\left. + \frac{1}{2}\hat{x}_i \hat{x}_j \frac{\partial^2}{\partial t'^2}\int d^3\vec{x}'\, x'^i x'^j T^{00}(\vec{x}',t') + \cdots\right]. \tag{2.64}$$

From the definitions of the moments and the relations implied by the conservation law, we have for J^{00}, out to second moments,

$$J^{00}(\vec{x},t) = 4\varphi_0^{-1}\frac{1}{r}\left[M(t') + \hat{x}_i\dot{D}^i(t') + \frac{1}{2}\hat{x}_i\hat{x}_j\ddot{Q}^{ij}(t')\right]. \tag{2.65}$$

For J^{0i} we need only two terms of the expansion (2.56) in order to include moments to the second.

$$J^{0i}(\vec{x},t) = 4\varphi_0^{-1}\frac{1}{r}\left[\int d^3\vec{x}'\,T^{0i}(\vec{x}',t') + \hat{x}_k\frac{\partial}{\partial t'}\int d^3\vec{x}'\,x'^k T^{0i}(\vec{x}',t')\right]. \tag{2.66}$$

Relations (2.62) and (2.63) then give

$$J^{0i}(\vec{x},t) = 4\varphi_0^{-1}\frac{1}{r}\left[\dot{D}^i(t') + \frac{1}{2}\hat{x}_k\ddot{Q}^{ik}(t')\right]. \tag{2.67}$$

And for J^{ij} only one term of (2.56) is required, giving immediately

$$J^{ij}(\vec{x},t) = 2\varphi_0^{-1}\frac{1}{r}\ddot{Q}^{ij}(t'). \tag{2.68}$$

The conservation law also implies that [See Appendix A]

$$\dot{M} = 0 \quad \text{and} \quad \ddot{D}^k = 0\,. \tag{2.69}$$

Therefore, from (2.65), (2.67), and (2.68) we have

$$\dot{J}^{00} = 2\varphi_0^{-1}\frac{1}{r}\hat{x}_i\hat{x}_j\dddot{Q}^{ij} \tag{2.70}$$

$$\dot{J}^{0i} = 2\varphi_0^{-1}\frac{1}{r}\hat{x}_k\dddot{Q}^{ik} \tag{2.71}$$

$$\dot{J}^{ij} = 2\varphi_0^{-1}\frac{1}{r}\dddot{Q}^{ij}\,. \tag{2.72}$$

To evaluate (2.55), we require $\dot{J}^{\lambda\kappa}\dot{J}_{\lambda\kappa}$, which may be written

$$\dot{j}^{\lambda\kappa}\dot{j}_{\lambda\kappa} = \dot{J}^{00}\ \dot{J}_{00} + 2\,\dot{J}^{0i}\dot{J}_{0i} + \dot{J}^{ij}\dot{J}_{ij}\,. \tag{2.73}$$

The signs of J^{00} and J^{ij} are independent of the raising / lowering of the pair of indices, but

$$J_{0i} = \eta_{0\alpha}\eta_{i\beta}J^{\alpha\beta} = -J^{0i}\,. \tag{2.74}$$

When this and (2.70) - (2.72) are used in (2.73) we get

$$\dot{j}^{\lambda\kappa}\dot{j}_{\lambda\kappa} = 4\varphi_0^{-2}\,\frac{1}{r^2}\left[\left(\hat{x}_i\,\hat{x}_j\,\dddot{Q}^{ij}\right)^2 - 2\left(\hat{x}_k\,\dddot{Q}^{ik}\right)\left(\hat{x}_j\,\dddot{Q}^{ij}\right) + \left(\dddot{Q}^{ij}\dddot{Q}_{ij}\right)\right]. \tag{2.75}$$

In completely analogous manner, we find

$$\dot{j}^{\lambda}{}_{\lambda} = \dot{J}^{0}{}_{0} + \dot{J}^{i}{}_{i} = -\dot{J}^{00} + \dot{J}^{i}{}_{i}\,, \tag{2.76}$$

so that from (2.70) - (2.72) we get

$$\dot{j}^{\lambda}{}_{\lambda} = -2\varphi_0^{-1}\,\frac{1}{r}\left(\hat{x}_i\,\hat{x}_j\,\dddot{Q}^{ij} - \dddot{Q}^{k}{}_{k}\right), \tag{2.77}$$

and finally

$$\left(\dot{j}^{\lambda}{}_{\lambda}\right)^2 = 4\varphi_0^{-2}\,\frac{1}{r^2}\left[\left(\hat{x}_i\,\hat{x}_j\,\dddot{Q}^{ij}\right)^2 - 2\,\dddot{Q}^{k}{}_{k}\left(\hat{x}_i\,\hat{x}_j\,\dddot{Q}^{ij}\right) + \left(\dddot{Q}^{k}{}_{k}\right)^2\right]. \tag{2.78}$$

When (2.75) and (2.78) are put into (2.55), and (2.16) is used again to eliminate φ_0 , we find

$$\left\langle t_{\mu\nu}\right\rangle = \frac{k_\mu k_\nu}{8\pi}\frac{G}{r^2}\frac{3+2\omega}{2+\omega}\left\langle\frac{2\omega^2+8\omega+7}{2(3+2\omega)^2}\left(\hat{x}_i\,\hat{x}_j\,\dddot{Q}^{ij}\right)^2 - \right.$$

$$\left. -\left(\hat{x}_k\,\dddot{Q}^{ik}\right)\left(\hat{x}_j\,\dddot{Q}^{ij}\right) + \frac{1}{2}\left(\dddot{Q}^{ij}\dddot{Q}_{ij}\right) + \left(\frac{1+\omega}{3+2\omega}\right)^2\dddot{Q}^{k}{}_{k}\left(2\,\hat{x}_i\,\hat{x}_j\,\dddot{Q}^{ij} - \dddot{Q}^{k}{}_{k}\right)\right\rangle. \tag{2.79}$$

Total Power

We now use (2.79) for $\langle t_{\mu\nu} \rangle$ in (2.45) and integrate over all directions in order to compute the total average flux of energy due to the tensor wave,

$$\left\langle \frac{dE}{dt} \right\rangle_{\substack{\text{total}\\ \text{(tensor)}}} = r^2 \int d\Omega\, \hat{x}^i \left\langle t^{0i} \right\rangle. \tag{2.80}$$

Note that

$$\hat{x}^i \left\langle t^{0i} \right\rangle = \hat{x}^i k^0 k^i [\cdots] = \hat{x}^i (-1)(-\hat{x}^i)[\cdots] = [\cdots],$$

which simplifies the evaluation of (2.80). Integration over all directions is accomplished readily with the help of

$$\int d\Omega\, \hat{x}_i \hat{x}_j = \frac{4\pi}{3} \delta_{ij} \tag{2.81}$$

and

$$\int d\Omega\, \hat{x}_i \hat{x}_j \hat{x}_l \hat{x}_m = \frac{4\pi}{15} \left(\delta_{ij}\delta_{lm} + \delta_{il}\delta_{jm} + \delta_{im}\delta_{jl} \right). \tag{2.82}$$

The result is:

$$\left\langle \frac{dE}{dt} \right\rangle_{\substack{\text{total}\\ \text{(tensor)}}} = \frac{G}{60} \frac{1}{(3+2\omega)(2+\omega)} \left\langle \left(24\omega^2 + 76\omega + 59\right)\left(\dddot{Q}^{ij}\right)^2 - \right.$$

$$\left. -\left(8\omega^2 + 12\omega + 3\right)\left(\dddot{Q}^k{}_k\right)^2 \right\rangle \tag{2.83}$$

We have already noted that the Brans-Dicke theory goes over to general relativity when $\omega \rightarrow \infty$. In this limit, (2.83) becomes

$$\left\langle \frac{dE}{dt} \right\rangle_{\substack{\text{total}\\ \text{GR}}} = \frac{G}{15} \left\langle 3\left(\dddot{Q}^{ij}\right)^2 - \left(\dddot{Q}^k{}_k\right)^2 \right\rangle, \tag{2 .84}$$

which is recognized as the result familiar from general relativity (14). It is customary to define the traceless tensor

$$D^{ij} \equiv 3Q^{ij} - \delta^{ij} Q^{k}{}_{k} \tag{2.85}$$

in terms of which (2.84) becomes

$$\left\langle \frac{dE}{dt} \right\rangle_{\substack{\text{total}\\ \text{GR}}} = \frac{G}{45}\left(\dddot{D}^{ij}\right)^2 . \tag{2.86}$$

Unfortunately, a comparable simplification does not follow if (2.85) is used in the result for the Brans-Dicke theory (2.83).

Energy-Momentum of the Gravitational Field - Scalar Part

To complete the analysis in the Brans-Dicke theory, we must carry through the analogous steps for the flux of energy carried by the scalar field. The weak field approximation of the tensor $T_{(\varphi)\mu\nu}$ associated with the scalar field is found by applying (2.5) and (2.26) to (2.4). This gives

$$T_{(\varphi)\mu\nu} = \frac{\omega}{8\pi}\varphi_0^{-1}\left(k_\mu k_\nu \dot{\psi}^2 - \frac{1}{2}\eta_{\mu\nu}k_\rho k^\rho \dot{\psi}^2\right) + \\ + \frac{1}{8\pi}\left(k_\mu k_\nu \ddot{\psi} - \eta_{\mu\nu}k_\rho k^\rho \ddot{\psi}\right). \tag{2.87}$$

The second and fourth terms above drop out in the far field due to (2.32) leaving

$$T_{(\varphi)\mu\nu} = \frac{k_\mu k_\nu}{8\pi}\left(\omega\varphi_0^{-1}\dot{\psi}^2 + \ddot{\psi}\right). \tag{2.88}$$

Recalling that ψ is small, one might be inclined to ignore the first term of (2.88) in favor of the second. However, consideration of the averaging process which is to be applied, as for the tensor field, reveals that it is the second, rather than the first, term which is appropriately ignored.

Consider

$$\left\langle \ddot{\psi} \right\rangle = \frac{1}{T}\int_0^T dt' \frac{d^2\psi}{dt'^2} = \frac{1}{T}\frac{d\psi}{dt'}\bigg|_0^T .$$

If $d\psi/dt'$ is a reasonably behaved function, $\langle\ddot{\psi}\rangle$ becomes arbitrarily small as T is increased. On the other hand,

$$\langle\dot{\psi}^2\rangle = \frac{1}{T}\int_0^T dt'\left(\frac{d\psi}{dt'}\right)^2$$

approaches a meaningful average as T is increased because of the positive definite periodic kernel. Therefore, for purposes of averaging, it is the first term of (2.88) that is to be retained.

$$\langle T_{(\varphi)\mu\nu}\rangle = \frac{k_\mu k_\nu}{8\pi}\omega\varphi_0^{-1}\langle\dot{\psi}^2\rangle. \tag{2.89}$$

Using (2.54) for $\dot{\psi}$, we find that

$$\langle T_{(\phi)\mu\nu}\rangle = \frac{k_\mu k_\nu}{8\pi}\omega\varphi_0\frac{1}{(3+2\omega)^2}\left\langle\left(\dot{j}^{\lambda}{}_{\lambda}\right)^2\right\rangle. \tag{2.90}$$

Finally, we use (2.78) for $\left(\dot{j}^{\lambda}{}_{\lambda}\right)^2$ and (2.16) for φ_0 to get

$$\langle T_{(\varphi)\mu\nu}\rangle = \frac{k_\mu k_\nu}{16\pi}\frac{G}{r^2}\frac{\omega}{(3+2\omega)(2+\omega)}\left\langle\left(\hat{x}_i\hat{x}_j\dddot{Q}^{ij}\right)^2 - \right.$$
$$\left. -\,2\,\dddot{Q}^k{}_k\left(\hat{x}_i\hat{x}_j\dddot{Q}^{ij}\right) + \left(\dddot{Q}^k{}_k\right)^2\right\rangle. \tag{2.91}$$

Note that the contribution from the scalar field goes to zero, as it should, as we make the transition to general relativity by letting ω go to infinity.

When $T_{(\varphi)\mu\nu}$ is substituted for $t_{\mu\nu}$ in (2.45) and then integrated over all directions, we find, with the help of (2.81) and (2.82), that

$$\left\langle\frac{dE}{dt}\right\rangle_{\substack{\text{total}\\ \text{(scalar)}}} = \frac{G}{10}\frac{\omega}{(3+2\omega)(2+\omega)}\left\langle\frac{1}{3}\left(\dddot{Q}^{ij}\right)^2 + \left(\dddot{Q}^k{}_k\right)^2\right\rangle. \tag{2.92}$$

The grand total flux due to both scalar and tensor quadrupole radiation is given by the sum of (2.83) and (2.92):

$$\left\langle \frac{dE}{dt} \right\rangle_{\text{total}} = \frac{G}{60} \frac{1}{(3+2\omega)(2+\omega)} \left\langle \left(24\omega^2 + 78\omega + 59\right)\left(\dddot{Q}^{ij}\right)^2 - \right.$$
$$\left. -\left(8\omega^2 + 6\omega + 3\right)\left(\dddot{Q}^k{}_k\right)^2 \right\rangle. \tag{2.93}$$

Application to a Binary System

In order to evaluate expressions for the flux such as (2.83), (2.92), or (2.93) it is necessary to form explicit expressions for

$$\left\langle \dddot{Q}^{ij} \dddot{Q}_{ij} \right\rangle$$

and

$$\left\langle \left(\dddot{Q}^k{}_k\right)^2 \right\rangle$$

for the system under consideration. For a binary system, we shall assume that in the first approximation the motion is Keplerian. Therefore averaging will be over one orbital period. We will first compute the averages that apply to a single body of the pair. It is then a simple matter of scaling to adapt the results to the system of two bodies.

For a point mass m, $Q^{ij}(t)$ defined by (2.59) is

$$Q^{ij}(t) \equiv \int d^3\vec{x}\, x^i x^j T^{00}(\vec{x}, t) =$$
$$= \iiint d\underline{x}^1 d\underline{x}^2 d\underline{x}^3\, m\, \underline{x}^i \underline{x}^j \delta\left(\underline{x}^1 - x^1(t)\right)\delta\left(\underline{x}^2 - x^2(t)\right)\delta\left(\underline{x}^3 - x^3(t)\right) \tag{2.94}$$

where $\underline{x}$ is the integration variable and x is the position of the mass. In terms of the specific system PSR 1913+16, we shall define

$$\begin{gathered} m = \text{mass of pulsar}, \\ M = \text{mass of companion, and} \\ \mu = \frac{GM^3}{(M+m)^2}. \end{gathered} \tag{2.95}$$

μ will be used to account for the fact that m may not be small when compared with M.

Since the orbit is Keplerian and, hence, planar, we can pick the coordinate system so that x^3 is zero. Then (2.94) reduces to

$$Q^{ij}(t) = 0 \qquad \text{(for } i \text{ or } j = 3) \tag{2.96}$$

$$Q^{11}(t) = m\left(x^1(t)\right)^2, \qquad Q^{22}(t) = m\left(x^2(t)\right)^2 \tag{2.97}$$

$$Q^{12}(t) = Q^{21}(t) = m\, x^1(t)\, x^2(t)\,. \tag{2.98}$$

As is well known, the position in the orbit plane is a transcendental function of the time. We shall, therefore, work with the parametric representation of the motion (15)

$$\begin{gathered} r = a\left(1-\varepsilon\cos E\right), \qquad t = \sqrt{\frac{a^3}{\mu}}(E-\varepsilon\sin E) \\ x^1(E) = a\left(\cos E-\varepsilon\right), \qquad x^2(E) = a\left(1-\varepsilon^2\right)^{1/2}\sin E \end{gathered} \tag{2.99}$$

where r = orbital radius to m, a = semimajor axis of the orbit of m, ε = eccentricity, E = eccentric anomaly. Over one orbit, E ranges from 0 to 2π. Since E, rather than t, will be used to locate the body in its orbit, we must re-write the expression for the time-average of a function so that it involves integration on dE.

$$\left\langle f\right\rangle \equiv \frac{1}{T}\int_0^T dt\; f(t)\,. \tag{2.100}$$

For a Keplerian orbit T has the value

$$T = 2\pi\sqrt{\frac{a^3}{\mu}}\,. \tag{2.101}$$

From (2.99) and (2.101) we have

$$t = \frac{T}{2\pi}(E - \varepsilon \sin E),$$

$$dt = \frac{T}{2\pi}(1 - \varepsilon \cos E)\, dE \; . \tag{2.102}$$

Therefore, if $f(t) = g\big(E(t)\big)$, we may write (2.100) as

$$\big\langle f(t) \big\rangle = \frac{1}{2\pi} \int_0^{2\pi} g(E)(1 - \varepsilon \cos E)\, dE \; . \tag{2.103}$$

Accordingly, we will first express the required contractions of $\dddot{Q}^{ij}$ in terms of the eccentric anomaly and will then use (2.103) to compute the time average over one orbital period.

Note incidentally that in what follows we can raise/lower all space indices without regard for sign changes because $\eta^{ij} = \delta^{ij}$.

$\underline{\left\langle \left(\dddot{Q}^k{}_k \right)^2 \right\rangle}$

From (2.97) and (2.99) we can write

$$Q^k{}_k = Q^1{}_1 + Q^2{}_2 = m\left[\left(x^1\right)^2 + \left(x^2\right)^2 \right] = m\, r^2$$

or

$$Q^k{}_k(E) = m\, a^2 (1 - \varepsilon \cos E)^2 \; . \tag{2.104}$$

Using (2.102), we find that all time derivatives are equivalently

$$\frac{d}{dt} = \frac{dE}{dt}\frac{d}{dE} = \frac{2\pi}{T}(1 - \varepsilon \cos E)^{-1} \frac{d}{dE} \, . \tag{2.105}$$

We find, therefore, that the derivatives of $Q^k{}_k$ are

$$\dot{Q}^k{}_k = 2\,m\,a^2\varepsilon\left(\frac{2\pi}{T}\right)\sin E \ ,$$

$$\ddot{Q}^k{}_k = 2\,m\,a^2\varepsilon\left(\frac{2\pi}{T}\right)^2(1-\varepsilon\cos E)^{-1}\cos E \ ,$$

$$\dddot{Q}^k{}_k = -2\,m\,a^2\varepsilon\left(\frac{2\pi}{T}\right)^3\frac{\sin E}{(1-\varepsilon\cos E)^3} =$$

$$= -2\,m\,\varepsilon\frac{\mu^{3/2}}{a^{5/2}}\frac{\sin E}{(1-\varepsilon\cos E)^3}\,, \tag{2.106}$$

from which

$$\left(\dddot{Q}^k{}_k\right)^2 = 4m^2\varepsilon^2\,\frac{\mu^3}{a^5}\frac{\sin^2 E}{(1-\varepsilon\cos E)^6}\,. \tag{2.107}$$

From (2.103) we find that

$$\left\langle\left(\dddot{Q}^k{}_k\right)^2\right\rangle = \frac{4}{\pi}m^2\varepsilon^2\,\frac{\mu^3}{a^5}\int_0^\pi\frac{\sin^2 E}{(1-\varepsilon\cos E)^5}dE \tag{2.108}$$

where we have used also the symmetry of the integrand over the range of integration.

In Appendix B we show that

$$\int_0^\pi\frac{\sin^2 E}{(1-\varepsilon\cos E)^5}dE = \frac{\pi}{8}\frac{4+\varepsilon^2}{\left(1-\varepsilon^2\right)^{7/2}}\,, \tag{2.109}$$

so that we have finally from (2.108) that

$$\left\langle\left(\dddot{Q}^k{}_k\right)^2\right\rangle = \frac{1}{2}m^2\,\frac{\mu^3}{a^5}\frac{\left(4+\varepsilon^2\right)\varepsilon^2}{\left(1-\varepsilon^2\right)^{7/2}}\,. \tag{2.110}$$

$\left\langle \dddot{Q}^{ij}\dddot{Q}_{ij} \right\rangle$

From the properties (2.96) - (2.98) of Q^{ij} we can write

$$\dddot{Q}^{ij}\dddot{Q}_{ij} = \left(\dddot{Q}^{11}\right)^2 + 2\left(\dddot{Q}^{12}\right)^2 + \left(\dddot{Q}^{22}\right)^2 . \tag{2.111}$$

We take each of the terms in order. First, from (2.97) and (2.99) we have that

$$Q^{11}(E) = m\,a^2(\cos E - \varepsilon)^2 . \tag{2.112}$$

Using (2.105) to compute the various derivatives we find

$$\dot{Q}^{11} = -2\,m\,a^2\left(\frac{2\pi}{T}\right)\frac{\sin E\,(\cos E - \varepsilon)}{1-\varepsilon\cos E},$$

$$\ddot{Q}^{11} = -2\,m\,a^2\left(\frac{2\pi}{T}\right)^2(1-\varepsilon\cos E)^{-3}\left(2\cos^2 E - \varepsilon\cos E - \varepsilon\cos^3 E + \varepsilon^2 - 1\right),$$

and

$$\begin{aligned}\dddot{Q}^{11} = -2\,m\,a^2\left(\frac{2\pi}{T}\right)^3(1-\varepsilon\cos E)^{-5}\sin E\,\big(\varepsilon\cos^2 E\, + \\ +2\varepsilon^2\cos E - 4\cos E - 3\varepsilon^3 + 4\varepsilon\big).\end{aligned} \tag{2.113}$$

Likewise, from (2.97) and (2.99) we have

$$Q^{22}(E) = m\,a^2\left(1-\varepsilon^2\right)\sin^2 E\ , \tag{2.114}$$

which leads to the derivatives

$$\dot{Q}^{22} = 2\,m\,a^2\left(\frac{2\pi}{T}\right)\left(1-\varepsilon^2\right)(1-\varepsilon\cos E)^{-1}\sin E\,\cos E\ ,$$

$$\ddot{Q}^{22} = 2\,m\,a^2\left(\frac{2\pi}{T}\right)^2\left(1-\varepsilon^2\right)(1-\varepsilon\cos E)^{-3}\left(\cos^2 E - \sin^2 E - \varepsilon\cos^3 E\right),$$

$$\dddot{Q}^{22} = 2\,m\,a^2 \left(\frac{2\pi}{T}\right)^3 \left(1-\varepsilon^2\right)\left(1-\varepsilon\cos E\right)^{-5} \sin E \left(3\varepsilon - 4\cos E + \varepsilon\cos^2 E\right). \tag{2.115}$$

And finally, from (2.98) and (2.99) we have

$$Q^{12} = Q^{21} = m\,a^2 \left(1-\varepsilon^2\right)^{1/2} \sin E \left(\cos E - \varepsilon\right), \tag{2.116}$$

which leads to the derivatives

$$\dot{Q}^{12} = m\,a^2 \left(\frac{2\pi}{T}\right)\left(1-\varepsilon^2\right)^{1/2}\left(1-\varepsilon\cos E\right)^{-1}\left(2\cos^2 E - \varepsilon\cos E - 1\right),$$

$$\ddot{Q}^{12} = m\,a^2 \left(\frac{2\pi}{T}\right)^2 \left(1-\varepsilon^2\right)^{1/2}\left(1-\varepsilon\cos E\right)^{-3} \sin E \left(2\varepsilon\cos^2 E - 4\cos E + 2\varepsilon\right),$$

$$\dddot{Q}^{12} = 2\,m\,a^2 \left(\frac{2\pi}{T}\right)^3 \left(1-\varepsilon^2\right)^{1/2}\left(1-\varepsilon\cos E\right)^{-5}\left(\varepsilon^2\cos^2 E + {}\right.$$
$$\left. + 3\varepsilon\cos E + \varepsilon\cos^3 E - 3\varepsilon^2 - 4\cos^2 E + 2\right). \tag{2.117}$$

When results (2.113), (2.115), and (2.117), together with (2.101) for T, are used in (2.111), one finds after considerable algebra that

$$\dddot{Q}^{ij}\dddot{Q}_{ij} = 4\,m^2 \frac{\mu^3}{a^5}\left(1-\varepsilon\cos E\right)^{-6}\left[8\left(1-\varepsilon^2\right) + \varepsilon^2\sin^2 E\right]. \tag{2.118}$$

On substitution of (2.118) into (2.103) for the average there results

$$\left\langle \dddot{Q}^{ij}\dddot{Q}_{ij} \right\rangle = \frac{4}{\pi} m^2 \frac{\mu^3}{a^5} \int_0^{\pi} \frac{8\left(1-\varepsilon^2\right) + \varepsilon^2 \sin^2 E}{\left(1-\varepsilon\cos E\right)^5}\, dE\ . \tag{2.119}$$

Note that the second term of the above is identically (2.108), which has the value given by (2.110). The first term is evaluated using[5]

$$\int_0^{\pi} \frac{dE}{(1-\varepsilon \cos E)^5} = \frac{\pi}{\left(1-\varepsilon^2\right)^{5/2}} P_4\left(\frac{1}{\sqrt{1-\varepsilon^2}}\right)$$

$$P_4(x) = \frac{1}{8}\left(35\, x^4 - 30\, x^2 + 3\right),$$

which leads to

$$\int_0^{\pi} \frac{dE}{(1-\varepsilon \cos E)^5} = \frac{\pi}{8} \frac{3\,\varepsilon^4 + 24\,\varepsilon^2 + 8}{\left(1-\varepsilon^2\right)^{9/2}}. \qquad (2.120)$$

Consequently, the first term of (2.119) is

$$4\, m^2 \frac{\mu^3}{a^5} \frac{3\,\varepsilon^4 + 24\,\varepsilon^2 + 8}{\left(1-\varepsilon^2\right)^{7/2}},$$

and the complete evaluation of (2.119) using also (2.110) is

$$\left\langle \dddot{Q}^{ij} \dddot{Q}_{ij} \right\rangle = \frac{1}{2} m^2 \frac{\mu^3}{a^5} \frac{25\,\varepsilon^4 + 196\,\varepsilon^2 + 64}{\left(1-\varepsilon^2\right)^{7/2}}. \qquad (2.121)$$

Scaling

Results (2.110) and (2.121) apply to the motion of one body m in a Keplerian orbit about a second body M. If used as is in the radiation formula (2.93) they will account for the energy loss due to the motion of m, but not of M. The overall loss rate due to the motion of both bodies is found from a simple scaling argument.

Let subscript 1 denote the pulsar m and 2 the companion M and, as above, let the origin of coordinates be at the barycenter. This condition is defined by

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[5] Ref. (16), entry 3.661 4 with $a = 1$, $b = -\varepsilon$, $n = 4$, and $a > |b|$.

$$m\, x_1^i + M\, x_2^i = 0$$

or,

$$x_2^i = -\frac{m}{M} x_1^i \,. \tag{2.122}$$

The overall moment Q^{ij} for the system consisting of both m and M is

$$Q^{ij} = m\, x_1^i x_1^j + M\, x_2^i x_2^j = \frac{m}{M}(m + M)\, x_1^i x_1^j \,, \tag{2.123}$$

where we have used (2.122) to express the moment in terms of the motion of m. This differs from (2.97) and (2.98) only by the factor of $(m + M)/M$. Therefore, results (2.110) and (2.121) can be made to apply to the system of two bodies of masses m and M by simply scaling by the factor $\left((m + M)/M\right)^2$. When this is done, and the definition (2.95) of μ is introduced, we have for the binary system

$$\left\langle \left(\dddot{Q}^k{}_k\right)^2 \right\rangle = \frac{G^3 m^2 M^7}{2\, a^5 (m + M)^4} \frac{\left(4 + \varepsilon^2\right)\varepsilon^2}{\left(1 - \varepsilon^2\right)^{7/2}} \tag{2.124}$$

and,

$$\left\langle \dddot{Q}^{ij} \dddot{Q}_{ij} \right\rangle = \frac{G^3 m^2 M^7}{2\, a^5 (m + M)^4} \frac{25\,\varepsilon^4 + 196\,\varepsilon^2 + 64}{\left(1 - \varepsilon^2\right)^{7/2}} \,. \tag{2.125}$$

For the special case of general relativity, these two results in (2.84) give

$$\left\langle \frac{dE}{dt} \right\rangle_{\substack{\text{total} \\ \text{GR}}} = \frac{32}{5} \frac{G^4 m^2 M^7}{a^5 (m + M)^4} \left[\frac{1 + \frac{73}{24}\varepsilon^2 + \frac{37}{96}\varepsilon^4}{\left(1 - \varepsilon^2\right)^{7/2}} \right]. \tag{2.126}$$

Effect on Orbital Period

One does not measure dE/dt directly but rather the change of orbital period T induced by dE/dt. Therefore, in order to apply our results to observation we need as a final step to relate dE/dt to dT/dt.

The semimajor axis of the relative orbit is

$$a' = \frac{m + M}{M} a$$

where it is to be recalled that a is the semimajor axis of the pulsar orbit. The total energy E of a Keplerian binary system is

$$E = -\frac{G\,m\,M}{2\,a'} = -\frac{G\,m\,M^2}{2\,a\,(m + M)} \tag{2.127}$$

from which

$$a = -\frac{G\,m\,M^2}{2\,(m + M)} E^{-1}. \tag{2.128}$$

The orbital period T can be related to the energy E by combining (2.101), (2.95) for μ and (2.128). The result is

$$T = \pi G \left(\frac{m^3 M^3}{2\,(m + M)} \right)^{1/2} (-E)^{-3/2}. \tag{2.129}$$

By taking the time derivative and afterward re-introducing (2.127) to restore the parameter a, we find that

$$\frac{dT}{dt} = \dot{T} = 6\pi \frac{(m + M)^2}{m} \sqrt{\frac{a^5}{G^3 M^7}} \frac{dE}{dt}. \tag{2.130}$$

Numerical Results for PSR 1913+16

We now use the published numerical values for the specific example of PSR 1913+16 to compute values for (2.124) and (2.125). These will be used in (2.93) to evaluate $\langle dE/dt \rangle$ from which dT/dt can be estimated using (2.130).

We take the following approximate values from Taylor *et al.* (1) for PSR 1913+16:

$$m = 1.39\, M_{sun} = 2.765 \text{ x } 10^{33} \text{ g}$$
$$M = 1.44\, M_{sun} = 2.864 \text{ x } 10^{33} \text{ g}$$
$$\sin i = 0.81$$
$$a \sin i = 2.3424 \text{ light-sec} = 7.0225 \text{ x } 10^{10} \text{ cm}$$
$$a = 8.67 \text{ x } 10^{10} \text{ cm}$$
$$\varepsilon = 0.617155$$

Also we use:

$$G = 6.673 \text{ x } 10^{-8} \text{ dyn cm}^2 \text{ g}^{-2}$$
$$c = 2.998 \text{ x } 10^{10} \text{ cm sec}^{-1}$$

Note that for dimensional reasons we have to restore a factor of c^5 to the denominator of dE/dt (2.93).

First, we find from (2.130) that for PSR 1913+16

$$\dot{T} = 2.2061 \text{ x } 10^{-44} \frac{\sec^3}{\text{g cm}^2} \frac{dE}{dt}. \tag{2.131}$$

From (2.124) and (2.125) we get

$$\left\langle \left(\dddot{Q}^k{}_k \right)^2 \right\rangle = 3.26 \text{ x } 10^{90} \left(\frac{\text{g cm}^2}{\sec^3} \right)^2 \tag{2.132}$$

and

$$\left\langle \dddot{Q}^{ij} \dddot{Q}_{ij} \right\rangle = 2.78 \text{ x } 10^{92} \left(\frac{\text{g cm}^2}{\sec^3} \right)^2 . \tag{2.133}$$

The average total radiation rate for the binary system in the Brans-Dicke theory is found from (2.93) and the above to be

$$\left\langle \frac{dE}{dt} \right\rangle = \frac{3.05 \text{ x } 10^{32}\omega^2 + 9.94 \text{ x } 10^{32}\omega + 7.53 \text{ x } 10^{32}}{(3+2\omega)(2+\omega)} \frac{\text{g cm}^2}{\text{sec}^3}. \qquad (2.134)$$

When this is used in (2.131) for the effect on the orbital period, we find

$$\dot{T} = \frac{6.73 \text{ x } 10^{-12}\omega^2 + 2.19 \text{ x } 10^{-11}\omega + 1.66 \text{ x } 10^{-11}}{(3+2\omega)(2+\omega)} \frac{\text{sec}}{\text{sec}}, \qquad (2.135)$$

where a positive result indicates a decreasing orbital period. In the limit $\omega \to \infty$, this goes over to the general relativistic prediction

$$\dot{T} = 3.36 \text{ x } 10^{-12} \frac{\text{sec}}{\text{sec}}. \qquad \text{(Gen. Rel.)} \qquad (2.136)$$

Figure l is a plot of $\dot{T}$ from (2.135) for PSR 1913+16 as a function of the free parameter

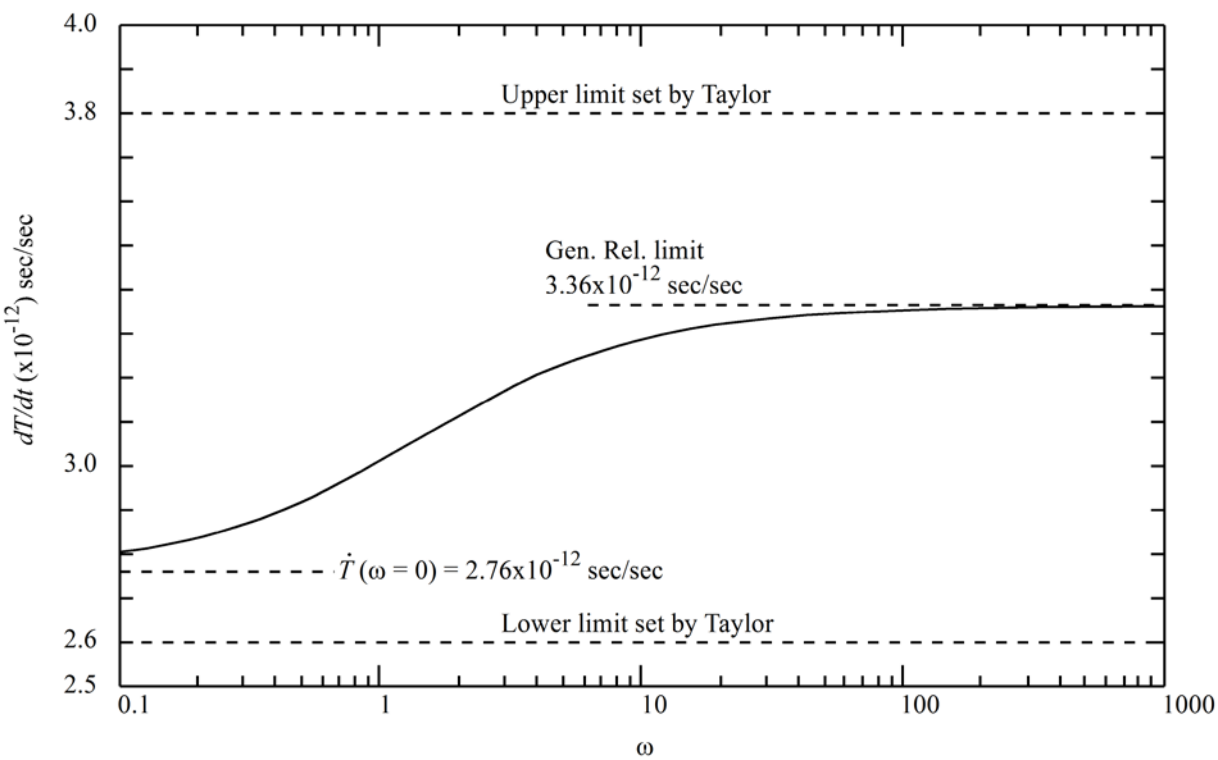

Figure 1. Orbital decay rate for PSR 1913+16 in the Brans-Dicke theory.

ω. Also indicated are the observational limits on $\dot{T}$ set by Taylor *et al.*, and the limiting general relativistic value (2.136). It is clear that the prediction for all positive values of ω is comfortably within the observational limits. Consequently, it is not possible to validate the Brans-Dicke theory or to establish bounds on ω from current observations of the orbital decay rate of PSR 1913+16 under the assumption of gravitational radiation in the quadrupole mode.

3.0 **Gravitational Radiation from Colliding Bodies in Brans-Dicke Theory**

General Results

We turn now to the problem of computing the gravitational radiation associated with a system of bodies in collision in the weak field limit of the Brans-Dicke theory. The results will be useful as a basis for studying other systems such as a hot gas in thermal equilibrium.

Spectral Representation of the Fields

Since the motion of the system is not periodic, the averaging technique that we applied in the previous section cannot be utilized. We assume, as before, that very far from the source ψ and $h_{\mu\nu}$ are functions of the scalar t'

$$t' = t - r = k_\lambda x^\lambda \tag{3.1}$$

$$k_0 = -k^0 = +1, \qquad k_i = -\hat{x}_i \tag{3.2}$$

$$\hat{x}^i \equiv \frac{x^i}{r} \qquad \text{(these as before).} \tag{3.3}$$

We assume further that in the far field $h_{\mu\nu}$ can be represented by a continuous spectral distribution $e_{\mu\nu}$ such that

$$h_{\mu\nu}(\vec{x},t) = \int_{-\infty}^{+\infty} d\varpi \; e_{\mu\nu}(\vec{x},\varpi)\, \exp\left(i\varpi k_\lambda x^\lambda\right) \tag{3.4}$$

which implies that

$$e_{\mu\nu}(\vec{x},\varpi)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}dt'\,h_{\mu\nu}(\vec{x},t')\,\exp(-i\varpi t'). \tag{3.5}$$

We use the symbol ϖ for the frequency in order to distinguish it from the free parameter ω of the Brans-Dicke theory, and the spatial angular measure $d\Omega$.

The linearity of the field equations, (2.9) and (2.14), in the weak field limit assures us that we can represent the scalar field also by a continuous distribution of the same form as for $h_{\mu\nu}$. That is

$$\psi(\vec{x},t)=\int_{-\infty}^{+\infty}d\varpi\;a(\vec{x},\varpi)\exp\left(i\varpi k_{\lambda}x^{\lambda}\right) \tag{3.6}$$

and

$$a(\vec{x},\varpi)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}dt'\psi(\vec{x},t')\exp(-i\varpi t'). \tag{3.7}$$

Though now in the form of integrals over continuous distributions, ψ and $h_{\mu\nu}$ still have the functional dependency on t' assumed in expressions (2.43) and (2.88) for $t_{\mu\nu}$ and $T_{(\varphi)\mu\nu}$. Keeping (3.1) in mind, we can immediately express both tensors using (3.4) and (3.6). We find from (2.43) that

$$\begin{aligned}t_{\mu\nu}=\frac{k_{\mu}k_{\nu}}{8\pi}\iint_{-\infty}^{+\infty}d\varpi\, d\varpi'\,\Big\{\tfrac{1}{2}\varphi_{0}\left[(\varpi')^{2}+\tfrac{1}{2}\varpi\varpi'\right]e^{\lambda\kappa}(\vec{x},\varpi)e_{\lambda\kappa}(\vec{x},\varpi)-\\ -2(1+\omega)\varphi_{0}^{-1}\left[2(1+\omega)(\varpi')^{2}+(2+\omega)\varpi\varpi'\right]a(\vec{x},\varpi)a(\vec{x},\varpi')\Big\}\times\\ \times\exp\left(i(\varpi+\varpi')k_{\lambda}x^{\lambda}\right)\end{aligned} \tag{3.8}$$

and from (2.88) that

$$T_{(\varphi)\mu\nu}=-\frac{k_{\mu}k_{\nu}}{8\pi}\left[\omega\,\varphi_{0}^{-1}\iint_{-\infty}^{+\infty}d\varpi\;d\varpi'\;a(\vec{x},\varpi)a(\vec{x},\varpi')\exp\left(i(\varpi+\varpi')k_{\lambda}x^{\lambda}\right)+\right.$$

$$+\int_{-\infty}^{+\infty} d\varpi \;\; \varpi^2 \, a\left(\vec{x},\varpi\right) \exp\left(i\varpi k_\lambda x^\lambda\right)\Bigg]. \qquad (3.9)$$

Definition of the 2-sided Power Spectral Density Function

Because the distributions $e_{\mu\nu}\left(\vec{x},\varpi\right)$ and $a\left(\vec{x},\varpi\right)$ are continuous in ϖ, there are components of each that are arbitrarily close together in frequency. As a consequence, the longest periodicity, which determines the minimum averaging interval T, tends to infinity. Therefore, the averaging process that we used previously no longer has a practical meaning, but it is still defined in the mathematical sense.

If we were to carry out the (mathematical) averaging process, we would have to integrate the instantaneous dE/dt over all time giving E, the total energy emitted by the system. Since ψ and $h_{\mu\nu}$ are now related to a and $e_{\mu\nu}$ by transform pair relationships, (3.6), (3.7), and (3.4), (3.5), we should be able to define a new function $P(\varpi)$ that, when integrated over all frequency, also gives the total energy E. That is,

$$E = \int_{-\infty}^{+\infty} d\varpi \, P(\varpi). \qquad (3.10)$$

Of the infinity of functions $P(\varpi)$ that satisfy (3.10), the most natural choice follows, not surprisingly, from an attempt to integrate dE/dt over all time as would be required by the averaging process.

Consider first (2.44) for dE/dt in the tensor field. Integration over all time gives, for a particular direction $\hat{x}$,

$$dE = r^2 d\Omega \, \hat{x}^i \int_{-\infty}^{+\infty} dt' \, t^{0i}(t'). \qquad (3.11)$$

(dE appears on the left side because we have not yet integrated over all directions $\hat{x}$. Otherwise dE should be considered on the same footing as E in (3.10) for purposes of defining the power spectral density). Clearly, integration of $t_{\mu\nu}$ over all time is an essential step in our computation of the total energy radiated by a system.

Recalling (3.1), we see from (3.8) that integration of $t_{\mu\nu}$ over all time involves

$$\int_{-\infty}^{+\infty} dt' \exp\left(i(\varpi + \varpi')t'\right) = 2\pi\delta(\varpi + \varpi') . \qquad (3.12)$$

Therefore, one of the frequency integrals from (3.8), say that on ϖ', can be performed immediately, with the result that

$$\varpi' = -\varpi \qquad (3.13)$$

in the remaining ϖ integral. Inspection of (3.5) and (3.7) reveals that

$$e_{\mu\nu}(\vec{x}, -\varpi) = e^{*}_{\mu\nu}(\vec{x}, \varpi) \qquad (3.14)$$

and

$$a(\vec{x}, -\varpi) = a^{*}(\vec{x}, \varpi) , \qquad (3.15)$$

assuming $h_{\mu\nu}$ and ψ to be real fields, and "star" denotes complex conjugate. We have, consequently, that

$$\int_{-\infty}^{+\infty} dt'\, t_{\mu\nu}(t') =$$

$$\frac{k_\mu k_\nu}{4} \int_{-\infty}^{+\infty} d\varpi \left[\frac{1}{4} \varphi_0\, e^{\lambda\kappa}(\vec{x}, \varpi) e^{*}_{\lambda\kappa}(\vec{x}, \varpi) - 2\omega(1+\omega)\varphi_0^{-1} \left| a(\vec{x}, \varpi) \right|^2 \right] \varpi^2 . \qquad (3.16)$$

An analogous argument holds for the scalar field with $T_{(\varphi)\mu\nu}$ substituted for $t_{\mu\nu}$ in (3.11), but there is a minor difference introduced by the fact that the second term of (3.9) involves only a single integral. Integration of $T_{(\varphi)\mu\nu}$ over all time involves, from the second term of (3.9),

$$\int_{-\infty}^{+\infty} d\varpi\ \varpi^2 a(\vec{x}, \varpi) \int_{-\infty}^{+\infty} dt' \exp\left(i\varpi t'\right) =$$

$$= 2\pi \int_{-\infty}^{+\infty} d\varpi\ \varpi^2 a(\vec{x}, \varpi) \delta(\varpi) = 0 . \qquad (3.17)$$

It is noteworthy that in the averaging process used in the previous section it was also this term that dropped out. We have finally from (3.9) that

$$\int_{-\infty}^{+\infty} dt'\, T_{(\varphi)\mu\nu}(t') = \frac{k_\mu k_\nu}{4}\,\omega\,\varphi_0^{-1}\int_{-\infty}^{+\infty} d\varpi\,\varpi^2 \left|a\left(\vec{x},\varpi\right)\right|^2 . \tag{3.18}$$

With (3.16) or (3.18) substituted into (3.11) for the respective field, we see that we now have expressions for the total energy that are of the form of (3.10). Using $dP(\varpi)$ to indicate that we have not yet integrated over all directions, we identify by comparison with (3.10) that for the tensor field

$$dP(\varpi)_{\text{tensor}} = \frac{r^2 d\Omega}{4}\varpi^2\left[\frac{1}{4}\varphi_0\, e^{\lambda\kappa}\left(\vec{x},\varpi\right)e^*_{\lambda\kappa}\left(\vec{x},\varpi\right) - 2\,\omega(1+\omega)\varphi_0^{-1}\left|a\left(\vec{x},\varpi\right)\right|^2\right] \tag{3.19}$$

and for the scalar field,

$$dP(\varpi)_{\text{scalar}} = \frac{r^2 d\Omega}{4}\,\omega\,\varphi_0^{-1}\varpi^2\left|a\left(\vec{x},\varpi\right)\right|^2 . \tag{3.20}$$

We have used above also the fact that

$$\hat{x}^i k^0 k^i = \hat{x}^i(-1)\left(-\hat{x}^i\right) = 1 .$$

Relations (3.19) and (3.20) are the 2-sided power spectral density functions associated with the tensor and scalar waves for radiation in the direction $\hat{x}$. They are "2-sided" because they conform to $P(\varpi)$ in (3.10), which involves integration over both positive and negative frequencies.

In the previous section, it proved advantageous to re-express the averaged energy-momentum tensors in terms of the common function $J_{\mu\nu}\left(\vec{x},t\right)$ defined by (2.50). By analogy, it will be useful here to re-express (3.19) and (3.20) in terms of the spectral representation of $J_{\mu\nu}$, $j_{\mu\nu}\left(\vec{x},\varpi\right)$. That is, we let

$$J_{\mu\nu}\left(\vec{x},t\right) = \int_{-\infty}^{+\infty} d\varpi\;\; j_{\mu\nu}\left(\vec{x},\varpi\right)\exp\left(i\varpi k_\lambda x^\lambda\right), \tag{3.21}$$

which implies that

$$j_{\mu\nu}\left(\vec{x},\varpi\right)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}dt'\,J_{\mu\nu}\left(\vec{x},t'\right)\exp\left(-i\varpi t'\right). \tag{3.22}$$

It follows from (2.51), and the representations (3.4), (3.6), and (3.21), that

$$e_{\mu\nu}\left(\vec{x},\varpi\right)=j_{\mu\nu}\left(\vec{x},\varpi\right)+2\varphi_0^{-1}\eta_{\mu\nu}(1+\omega)\,a\left(\vec{x},\varpi\right), \tag{3.23}$$

and from (2.53) that

$$j^{\lambda}{}_{\lambda}\left(\vec{x},\varpi\right)=-2\varphi_0^{-1}(3+2\omega)\,a\left(\vec{x},\varpi\right). \tag{3..24}$$

From (3.23), we have

$$e^{\lambda\kappa}e^{*}_{\lambda\kappa}=j^{\lambda\kappa}j^{*}_{\lambda\kappa}+2\varphi_0^{-1}(1+\omega)\left(j^{\lambda}{}_{\lambda}\,a^{*}+j^{*\lambda}{}_{\lambda}\,a\right)+16\,\varphi_0^{-2}(1+\omega)^2|a|^2\,. \tag{3.25}$$

From (3.24), we find that

$$j^{\lambda}{}_{\lambda}\,a^{*}+j^{*\lambda}{}_{\lambda}\,a=-4\varphi_0^{-1}(3+2\omega)|a|^2\,, \tag{3.26}$$

so that (3.25) reduces to

$$e^{\lambda\kappa}e^{*}_{\lambda\kappa}=j^{\lambda\kappa}j^{*}_{\lambda\kappa}-8\varphi_0^{-2}(1+\omega)|a|^2\,. \tag{3.27}$$

From (3.24), we have also that

$$|a|^2=\frac{\left|j^{\lambda}{}_{\lambda}\right|^2}{4\varphi_0^{-2}(3+2\omega)^2}\,. \tag{3.28}$$

The last two results in (3.19) and (3.20) give us

$$dP(\varpi)_{\text{tensor}}=\frac{r^2d\Omega}{8}\varphi_0\varpi^2\left[\frac{1}{2}\,j^{\lambda\kappa}\left(\vec{x},\varpi\right)j^{*}_{\lambda\kappa}\left(\vec{x},\varpi\right)-\left(\frac{1+\omega}{3+2\omega}\right)^2\left|j^{\lambda}{}_{\lambda}\left(\vec{x},\varpi\right)\right|^2\right] \tag{3.29}$$

and,

$$dP(\varpi)_{\text{scalar}} = \frac{r^2 d\Omega}{16} \varphi_0 \frac{\omega}{(3+2\omega)^2} \varpi^2 \left| j^{\lambda}{}_{\lambda}(\vec{x},\varpi) \right|^2 . \tag{3.30}$$

Spectral Representation of the Motion

Since we have expressed the fields and the flux in the spectral representation, it is appropriate that we do so also for the source $T_{\mu\nu}$. Accordingly, we let

$$T_{\mu\nu}(\vec{x},t) = \int_{-\infty}^{+\infty} d\varpi \, \tilde{T}_{\mu\nu}(\vec{x},\varpi) \exp(i\varpi t) . \tag{3.31}$$

$J_{\mu\nu}$ defined by (2.50) becomes

$$J_{\mu\nu}(\vec{x},t) = 4\varphi_0^{-1} \int \frac{d^3\vec{x}'}{|\vec{x}'-\vec{x}|} \int_{-\infty}^{+\infty} d\varpi \tilde{T}_{\mu\nu}(\vec{x}',\varpi) \exp\left(i\varpi(t-|\vec{x}'-\vec{x}|)\right) \cong$$

$$\cong \int_{-\infty}^{+\infty} d\varpi \left[\frac{4\varphi_0^{-1}}{r} \int d^3\vec{x}' \, \tilde{T}_{\mu\nu}(\vec{x}',\varpi) \exp\left(i\varpi \hat{x}\cdot\vec{x}'\right) \right] \exp\left(i\varpi(t-r)\right), \tag{3.32}$$

where we have exchanged the order of the integrations and have used the approximations

$$|\vec{x}'-\vec{x}|^{-1} \cong 1/r \qquad \text{and} \qquad |\vec{x}'-\vec{x}| \cong r - \hat{x}\cdot\vec{x}'$$

for $r >> |\vec{x}'|$. By comparing (3.32) with (3.21), and recalling that $t' = k_\lambda x^\lambda$, we see that

$$j_{\mu\nu}(\vec{x},\varpi) = \frac{4\varphi_0^{-1}}{r} \hat{T}_{\mu\nu}(\hat{x},\varpi) \tag{3.33}$$

where:

$$\hat{T}_{\mu\nu}(\hat{x},\varpi) \equiv \int d^3\vec{x}' \, \tilde{T}_{\mu\nu}(\vec{x}',\varpi) \exp\left(i\varpi \, \hat{x}\cdot\vec{x}'\right) . \tag{3.34}$$

Using (3.33), we can express (3.29) and (3.30) in the far field as

$$dP(\varpi)_{\text{tensor}} = 2\, d\Omega\, \varphi_0^{-1} \varpi^2 \left[\frac{1}{2} \hat{T}^{\lambda\kappa}(\hat{x},\varpi) \hat{T}^*_{\lambda\kappa}(\hat{x},\varpi) - \left(\frac{1+\omega}{3+2\omega} \right)^2 \left| \hat{T}^{\lambda}{}_{\lambda}(\hat{x},\varpi) \right|^2 \right] \tag{3.35}$$

and,

$$dP(\varpi)_{\text{scalar}} = d\Omega\, \varphi_0^{-1} \frac{\omega}{(3+2\omega)^2} \varpi^2 \left| \hat{T}^{\lambda}{}_{\lambda}(\hat{x},\varpi) \right|^2 . \tag{3.36}$$

Application to a System of Colliding Particles

The results above apply generally to any problem in which the motion is aperiodic. To apply the foregoing to a specific problem, we need a functional form for $\tilde{T}_{\mu\nu}(\vec{x},\varpi)$. For the case of a system of colliding particles, we can find this function without formally utilizing the inverse transform associated with (3.31).

Consider a system of point particles with 4-momenta P_n^{μ} that collide at the origin at $t = 0$. (n identifies the particle). Assume that the particles are conserved and that after the collision they have momenta $P_n'^{\mu}$. The energy-momentum tensor of the system is

$$T_{\mu\nu}(\vec{x},t) = \sum_n \frac{P_n^{\mu} P_n^{\nu}}{E_n} \delta^3(\vec{x} - \vec{v}_n t)\theta(-t) + \sum_n \frac{P_n'^{\mu} P_n'^{\nu}}{E_n'} \delta^3(\vec{x} - \vec{v}_n' t)\theta(+t) \tag{3.37}$$

where $\theta(t)$ is the step-function having the value

$$\left. \begin{aligned} \theta(t) &= 0 \\ \theta(t) &= 1 \end{aligned} \right\} \quad \begin{aligned} &(t > 0) \\ &(t < 0) \end{aligned} \tag{3.38}$$

and integral representation

$$\theta(t) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} d\varpi \frac{\exp(i\varpi t)}{\varpi - i\varepsilon} = \frac{-1}{2\pi i} \int_{-\infty}^{+\infty} d\varpi \frac{\exp(-i\varpi t)}{\varpi + i\varepsilon} . \tag{3.39}$$

where $\varepsilon = 0^+$. The 3-dimensional delta-function has the integral representation

$$\delta^3(\vec{x}) = \frac{1}{(2\pi)^3}\int d^3\vec{q}\,\exp\left(i\vec{q}\cdot\vec{x}\right). \tag{3.40}$$

$\vec{v}_n$ is the velocity of the n-th particle. E_n, the total relativistic energy of the n-th particle, is defined through

$$P_n^0 = E_n \qquad \text{and} \qquad \vec{P}_n = E_n\vec{v}_n . \tag{3.41}$$

If we substitute the integral representations (3.39) and (3.40) into (3.37) we get

$$T^{\mu\nu}(\vec{x},t) = \int_{-\infty}^{+\infty} d\varpi\,\frac{1}{(2\pi)^4 i}\left[-\sum_n \frac{P_n^\mu P_n^\nu}{E_n}\int d^3\vec{q}\,\frac{\exp\left(i\vec{q}\cdot\left(\vec{x}-\vec{v}_n t\right)\right)}{\varpi + i\varepsilon} + \right.$$
$$\left. + \sum_n \frac{P_n'^\mu P_n'^\nu}{E_n'}\int d^3\vec{q}\,\frac{\exp\left(i\vec{q}\cdot\left(\vec{x}-\vec{v}_n' t\right)\right)}{\varpi - i\varepsilon}\right]\exp\left(i\varpi t\right). \tag{3.42}$$

In the first integral in brackets above, let us make the change of variable $\varpi' = \varpi - \vec{q}\cdot\vec{v}_n$ and in the second $\varpi' = \varpi - \vec{q}\cdot\vec{v}_n'$. When this is done, and the prime subsequently dropped from ϖ', there results

$$T^{\mu\nu}(\vec{x},t) = \int_{-\infty}^{+\infty} d\varpi\,\frac{1}{(2\pi)^4 i}\left[-\sum_n \frac{P_n^\mu P_n^\nu}{E_n}\int d^3\vec{q}\,\frac{\exp\left(i\vec{q}\cdot\vec{x}\right)}{\varpi + \vec{q}\cdot\vec{v}_n + i\varepsilon} + \right.$$
$$\left. + \sum_n \frac{P_n'^\mu P_n'^\nu}{E_n'}\int d^3\vec{q}\,\frac{\exp\left(i\vec{q}\cdot\vec{x}\right)}{\varpi + \vec{q}\cdot\vec{v}_n' - i\varepsilon}\right]\exp\left(i\varpi t\right). \tag{3.43}$$

Comparison with (3.31) shows immediately that

$$\tilde{T}^{\mu\nu}(\vec{x},\varpi) = \frac{1}{(2\pi)^4 i}\left[-\sum_n \frac{P_n^\mu P_n^\nu}{E_n}\int d^3\vec{q}\,\frac{\exp\left(i\vec{q}\cdot\vec{x}\right)}{\varpi + \vec{q}\cdot\vec{v}_n + i\varepsilon} + \right.$$

$$+\sum_n \frac{P_n'^{\mu} P_n'^{\nu}}{E_n'} \int d^3\vec{q}\ \frac{\exp\left(i\vec{q}\cdot\vec{x}\right)}{\varpi + \vec{q}\cdot\vec{v}_n' - i\varepsilon}\Bigg]. \tag{3.44}$$

If we now apply (3.34) to (3.44) in order to evaluate $\hat{T}_{\mu\nu}$,we see that there arises

$$\int_{-\infty}^{+\infty} d^3\vec{x}'\ \exp\left(i\left(\vec{q} + \varpi\,\hat{x}\right)\cdot\vec{x}'\right) = (2\pi)^3\,\delta^3\left(\vec{q} + \varpi\,\hat{x}\right). \tag{3.45}$$

Consequently, the integrations on $\vec{q}$ from (3.44) can be carried out with the effect that $\vec{q} = -\varpi\,\hat{x}$. Thus (3.34) becomes

$$\hat{T}^{\mu\nu}\left(\hat{x},\varpi\right) = \frac{1}{2\pi i}\left[-\sum_n \frac{P_n^{\mu} P_n^{\nu}}{E_n} \frac{1}{\varpi\left(1 - \hat{x}\cdot\vec{v}_n\right) + i\varepsilon} + \right.$$

$$\left. + \sum_n \frac{P_n'^{\mu} P_n'^{\nu}}{E_n'} \frac{1}{\varpi\left(1 - \hat{x}\cdot\vec{v}_n'\right) - i\varepsilon}\right]. \tag{3.46}$$

Recall that $\hat{x}$ is a unit vector and, with $c = 1$, $\left|\vec{v}_n\right| < 1$ for a massive particle so that $1 - \hat{x}\cdot\vec{v}_n$ in the denominators above can never be zero. We may, therefore, now drop the $\pm i\varepsilon$.

To ease the notation, we recognize that

$$E_n\varpi\,\left(1 - \hat{x}\cdot\vec{v}_n\right) = \varpi\left(P_n^0\,k_0 + P_n^i\,k_i\right) = \varpi\,P_n^{\mu}k_{\mu} \tag{3.47}$$

using (3.2) and (3.41). As a further help, let us adopt the convention that N runs over particles in both the initial and final states and that

$$\left.\begin{aligned} \eta_N &= -1 \\ \eta_N &= +1 \end{aligned}\right\} \quad \begin{aligned} &N \text{ corresponding to initial state.} \\ &N \text{ corresponding to final state.} \end{aligned} \tag{3.48}$$

Then $\hat{T}^{\mu\nu}$ may be very compactly written as

$$\hat{T}^{\mu\nu}(\hat{x},\varpi) = \frac{1}{2\pi i}\sum_N \frac{P_N^{\mu} P_N^{\nu} \eta_N}{\varpi P_N^{\lambda} k_{\lambda}} . \tag{3.49}$$

In order to evaluate (3.35) and (3.36), we require also the trace of $\hat{T}^{\mu\nu}$. From the numerator of (3.49), this involves

$$P_N^{\lambda} P_{N\lambda} = -m_N^2 , \tag{3.50}$$

where m_N is the rest mass of the N - th colliding particle. Therefore,

$$\hat{T}^{\lambda}{}_{\lambda}(\hat{x},\varpi) = -\frac{1}{2\pi i}\sum_N \frac{m_N^2 \eta_N}{\varpi P_N^{\lambda} k_{\lambda}} . \tag{3.51}$$

Using (3.49) and (3.51) in (3.35) and (3.36), we find that

$$\frac{dP(\varpi)}{d\Omega}^{\text{tensor}} = \frac{G\,(3+2\omega)}{4\pi^2(2+\omega)} \sum_{N,M} \frac{\eta_N \eta_M}{\left(P_N^{\lambda} k_{\lambda}\right)\left(P_M^{\lambda} k_{\lambda}\right)} \left[\frac{\left(P_N^{\lambda} P_{M\lambda}\right)^2}{2} - \left(\frac{1+\omega}{3+2\omega}\right)^2 m_N^2 m_M^2 \right] \tag{3.52}$$

and,

$$\frac{dP(\varpi)}{d\Omega}^{\text{scalar}} = \frac{G}{8\pi^2} \frac{\omega}{(2+\omega)(3+2\omega)} \sum_{N,M} \frac{\eta_N \eta_M m_N^2 m_M^2}{\left(P_N^{\lambda} k_{\lambda}\right)\left(P_M^{\lambda} k_{\lambda}\right)} , \tag{3.53}$$

where we have used also (2.16) for φ_0. Note that both of the above are independent of frequency ϖ. This is because we have modeled the collision as being instantaneous. Note also that the scalar contribution (3.53) goes to zero as the free parameter ω of the Brans-Dicke theory is allowed to approach infinity, in which case the theory goes over to general relativity.

Total Power Spectral Density

We are of course interested also in the spectrum of the radiation associated with integration over all directions $\hat{x}$. To put (3.52) and (3.53) into a convenient form for integration on $d\Omega$, we introduce the relative speed of the particles N and M,

$$\beta_{NM} \equiv \left(1 - \frac{m_N^2\, m_M^2}{\left(P_N^{\lambda} P_{M\lambda}\right)^2}\right)^{1/2} \tag{3.54}$$

in terms of which

$$m_N^2\, m_M^2 = \left(P_N^{\lambda} P_{M\lambda}\right)^2 \left(1 - \beta_{NM}^2\right). \tag{3.55}$$

When this is introduced into (3.52) and (3.53), we have

$$\frac{dP}{d\Omega}^{\text{tensor}} = \frac{G\,(3+2\omega)}{4\pi^2 (2+\omega)} \sum_{N,M} \eta_N\, \eta_M \left[\frac{1}{2} - \left(\frac{1+\omega}{3+2\omega}\right)^2 \left(1 - \beta_{NM}^2\right)\right] \times \\ \times \frac{\left(P_N^{\lambda} P_{M\lambda}\right)^2}{\left(P_N^{\lambda} k_{\lambda}\right)\left(P_M^{\lambda} k_{\lambda}\right)} \tag{3.56}$$

and,

$$\frac{dP}{d\Omega}^{\text{scalar}} = \frac{G}{8\pi^2} \frac{\omega}{(2+\omega)(3+2\omega)} \sum_{N,M} \eta_N\, \eta_M \left(1 - \beta_{NM}^2\right) \times \\ \times \frac{\left(P_N^{\lambda} P_{M\lambda}\right)^2}{\left(P_N^{\lambda} k_{\lambda}\right)\left(P_M^{\lambda} k_{\lambda}\right)}. \tag{3.57}$$

Thus, in this form, the integrations of the tensor and the scalar components over $d\Omega$ each involve a common integral which is found to be [See Appendix C]

$$\int d\Omega \; \frac{\left(P_N^{\lambda} P_{M\lambda}\right)^2}{\left(P_N^{\lambda} k_{\lambda}\right)\left(P_M^{\lambda} k_{\lambda}\right)} = \frac{2\pi\, m_N\, m_M}{\left(1 - \beta_{NM}^2\right)^{1/2} \beta_{NM}} \ln\left(\frac{1 - \beta_{NM}}{1 + \beta_{NM}}\right). \tag{3.58}$$

It follows from (3.56) and (3.57) that

$$P(\varpi)^{\text{tensor}} = \frac{G\,(3+2\omega)}{2\pi(2+\omega)} \sum_{N,M} \eta_N\,\eta_M\,m_N\,m_M \left[\frac{\frac{1}{2} - \left(\frac{1+\omega}{3+2\omega}\right)^2 \left(1-\beta_{NM}^2\right)}{\left(1-\beta_{NM}^2\right)^{1/2} \beta_{NM}} \right] \times$$

$$\times \ln\left(\frac{1-\beta_{NM}}{1+\beta_{NM}}\right) \qquad (3.59)$$

and,

$$P(\varpi)^{\text{scalar}} = \frac{G}{4\pi} \frac{\omega}{(2+\omega)(3+2\omega)} \sum_{N,M} \eta_N\,\eta_M\,m_N\,m_M\, \frac{\left(1-\beta_{NM}^2\right)^{1/2}}{\beta_{NM}} \times$$

$$\times \ln\left(\frac{1-\beta_{NM}}{1+\beta_{NM}}\right). \qquad (3.60)$$

The grand total 2-sided power spectral density for a system of colliding particles in the Brans-Dicke theory is, from the sum of (3.59) and (3.60)

$$P(\varpi)^{\text{total}} = \frac{G}{4\pi} \frac{1}{(2+\omega)(3+2\omega)} \sum_{N,M} \eta_N\,\eta_M\,m_N\,m_M \times$$

$$\times \left[\frac{(2\omega+7)(\omega+1) + \left(2\omega^2+3\omega+2\right)\beta_{NM}^2}{\left(1-\beta_{NM}^2\right)^{1/2} \beta_{NM}} \right] \ln\left(\frac{1-\beta_{NM}}{1+\beta_{NM}}\right). \qquad (3.61)$$

It is to be noted that the multipole expansion of the motion has not been introduced into either the general results or the specific application of a system of colliding particles. This is because it has been assumed that the motion has a known spectral representation (3.31). In the case of the colliding particles, it proved possible to find the exact spectral representation so that our results are good to all multipole orders.

4.0 **Quadrupole Gravitational Radiation in the Rosen Theory**

We would like to investigate quadrupole gravitational radiation in the bi-metric theory of Rosen (4,5) both in general terms and as applied to the specific problems already treated in the Brans-Dicke theory: a binary system, PSR 1913+16 in particular, and a system of colliding particles.

Field Equations

Rosen's theory is characterized by two metric tensors: $g_{\mu\nu}$ that describes purely gravitational effects, as in general relativity, and $\gamma_{\mu\nu}$ that accounts for inertial effects independent of gravity. The field equations are, from (5),[6]

$$N_{\mu\nu} - \frac{1}{2} g_{\mu\nu} N^{\lambda}{}_{\lambda} = -8\pi\kappa G T_{\mu\nu}, \tag{4.1}$$

where

$$N_{\mu\nu} = \frac{1}{2} g_{\mu\nu|\alpha\underline{\alpha}} - \frac{1}{2} g^{\lambda\rho} g_{\mu\lambda|\alpha} g_{\nu\rho|\underline{\alpha}} \tag{4.2}$$

$$\kappa \equiv \left(g/\gamma\right)^{1/2} \tag{4.3}$$

$$g \equiv -\det.\left(g_{\mu\nu}\right), \qquad \gamma \equiv -\det.\left(\gamma_{\mu\nu}\right). \tag{4.4}$$

A bar under an index indicates that it is to be raised / lowered with the γ metric. A vertical bar | indicates that the indices to the right represent covariant derivatives with respect to $\gamma_{\mu\nu}$ – that is, in which the affine connection is formed from $\gamma_{\mu\nu}$.

The conservation law is, as in general relativity and the Brans-Dicke theory,

$$T^{\mu\nu}{}_{;\nu} = 0\,, \tag{4.5}$$

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[6] In (5), Rosen takes $G = 1$. We re-insert G here for compatibility with results already derived in the Brans-Dicke theory. c is still taken to be 1.

where semicolon represents covariant differentiation with respect to $g_{\mu\nu}$.

We will assume that there are no inertial forces so that we are free to take

$$\gamma_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag.}(-1,1,1,1). \tag{4.6}$$

Consequently, from (4.3) and (4.4) we have

$$\gamma = -\det.(\eta_{\mu\nu}) = 1 \tag{4.7}$$

and

$$\kappa = g^{1/2}. \tag{4.8}$$

A further consequence of (4.6) is that the affine connection formed from $\gamma_{\mu\nu}$ is zero so that all | symbols go over to ordinary partial derivatives. Therefore, the field equations (4.1) and (4.2) go over to

$$N_{\mu\nu} - \frac{1}{2} g_{\mu\nu} N^{\lambda}{}_{\lambda} = -8\pi g^{1/2} G\, T_{\mu\nu} \tag{4.9}$$

and

$$N_{\mu\nu} = \frac{1}{2} g_{\mu\nu,\alpha\underline{\alpha}} - \frac{1}{2} g^{\lambda\rho} g_{\mu\lambda,\alpha} g_{\nu\rho,\underline{\alpha}}. \tag{4.10}$$

Indices with a bar under them are, of course, now raised/lowered with the help of $\eta_{\mu\nu}$.

It will be helpful to have a compact expression for the trace of $N_{\mu\nu}$. From (4.10) we have

$$N^{\lambda}{}_{\lambda} = g^{\lambda\tau} N_{\lambda\tau} = \frac{1}{2}\left(g^{\lambda\tau} g_{\lambda\tau,\alpha\underline{\alpha}} - g^{\lambda\tau} g^{\sigma\rho} g_{\lambda\sigma,\alpha} g_{\tau\rho,\underline{\alpha}}\right). \tag{4.11}$$

Since, by definition,

$$g^{\sigma\rho} g_{\tau\rho} = \delta^{\sigma}{}_{\tau}, \tag{4.12}$$

it follows that

$$g^{\sigma\rho} g_{\tau\rho,\underline{\alpha}} = -g^{\sigma\rho}{}_{,\underline{\alpha}} g_{\tau\rho}. \tag{4.13}$$

Therefore, the second term of (4.11) can be written

$$- g^{\lambda\tau} g^{\sigma\rho} g_{\lambda\sigma,\alpha} g_{\tau\rho,\underline{\alpha}} = + g^{\lambda\tau} g_{\tau\rho}\, g^{\sigma\rho}{}_{,\underline{\alpha}} g_{\lambda\sigma,\alpha} = g^{\lambda\sigma}{}_{,\underline{\alpha}} g_{\lambda\sigma,\alpha} \tag{4.14}$$

where, in the last step, we have used (4.12) and the symmetry of $g_{\mu\nu}$ with respect to interchange of indices. When (4.14) is used in (4.11), we find

$$N^{\lambda}{}_{\lambda} = \frac{1}{2}\left(g^{\lambda\sigma} g_{\lambda\sigma,\alpha\underline{\alpha}} + g^{\lambda\sigma}{}_{,\underline{\alpha}}\, g_{\lambda\sigma,\alpha} \right)$$

or,

$$N^{\lambda}{}_{\lambda} = \frac{1}{2}\left(g^{\lambda\sigma} g_{\lambda\sigma,\alpha} \right)_{,\underline{\alpha}}. \tag{4.15}$$

Let us define

$$A_{\alpha} \equiv g^{\lambda\sigma} g_{\lambda\sigma,\alpha}, \tag{4.16}$$

so that we can write the trace of $N_{\mu\nu}$ as

$$N^{\lambda}{}_{\lambda} = \frac{1}{2} A_{\alpha,\underline{\alpha}}. \tag{4.17}$$

<u>Gravitational Energy-Momentum Tensor</u>

The gravitational energy-momentum tensor $t_{\mu\nu}$ in the Rosen theory[7] is, with $\gamma_{\mu\nu} = \eta_{\mu\nu}$,

$$16\pi G\, g^{1/2} t_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} g^{\sigma\tau} g_{\lambda\sigma,\mu} g_{\rho\tau,\nu} - \frac{1}{4} A_{\mu} A_{\nu} - \eta_{\mu\nu} L_{g}, \tag{4.18}$$

where A_{μ} is defined by (4.16), and L_{g} is defined to be

$$L_{g} = \frac{1}{4} g^{\lambda\rho} g^{\sigma\tau} g_{\lambda\sigma,\alpha} g_{\rho\tau,\underline{\alpha}} - \frac{1}{8} A_{\alpha} A_{\underline{\alpha}}. \tag{4.19}$$

---

[7] In (4) and (5), Rosen uses $t_{\mu\nu}$ to denote the gravitational energy-momentum tensor *density*.

Weak Field Approximation

We go over to the weak field approximation by considering, as usual,

$$g_{\mu\nu} = \gamma_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \tag{4.20}$$

where $h_{\mu\nu}$ is small. (4.12) implies that

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}\,. \tag{4.21}$$

In the limit of very small $h_{\mu\nu}$, covariant derivatives with respect to $g_{\mu\nu}$ go over to ordinary partial derivatives because of the rightmost expression in (4.20). Therefore, in the weak field limit, all derivatives become partial derivatives and all indices are raised/lowered with $\eta_{\mu\nu}$.

The first, and most obvious, consequence is that the conservation law (4.5) becomes

$$T^{\mu\nu}{}_{,\nu} = 0\,. \tag{4.22}$$

When (4.20) and (4.21) are substituted into (4.16) and the lowest order term kept, we find that in the weak field limit

$$A_\alpha = \eta^{\lambda\sigma} h_{\lambda\sigma,\alpha} = h^{\lambda}{}_{\lambda,\alpha}\,. \tag{4.23}$$

Similarly, when (4.20) and (4.21) are used in the field equations (4.9) and (4.10), and (4.17) and (4.23) are used for $N^{\lambda}{}_{\lambda}$, we find to lowest order in $h_{\mu\nu}$

$$\frac{1}{2} h_{\mu\nu,\alpha\underline{\alpha}} - \frac{1}{4}\eta_{\mu\nu} h^{\lambda}{}_{\lambda,\alpha\underline{\alpha}} = -8\pi G\, T_{\mu\nu}$$

or,

$$\Box^2\left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h^{\lambda}{}_{\lambda} \right) = -16\pi G\, T_{\mu\nu}\,. \tag{4.24}$$

Attention is drawn to the fact that in the weak field limit

$$g^{1/2} = \sqrt{-\det.\left(\eta_{\mu\nu}\right)} = 1 \tag{4.25}$$

so that $g^{1/2}$ no longer appears next to the matter tensor $T_{\mu\nu}$ on the right hand side of the field equation. In the *Introduction* we stated that dipole radiation appears in the Rosen theory because the effective strength of the elements of $T_{\mu\nu}$ is determined by the factor $g^{1/2}$, which is generally position dependent. In the weak field approximation, $g^{1/2}$ goes to the constant 1 so that, in this approximation, there can be no dipole radiation in the Rosen theory.

The field equation (4.24), when contracted, gives

$$\Box^2 h^{\lambda}{}_{\lambda} = 16\pi G\, T^{\lambda}{}_{\lambda}\,. \tag{4.26}$$

When this is introduced back into (4.24) and taken to the right side, we have the equivalent field equation

$$\Box^2 h_{\mu\nu} = -16\pi G\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T^{\lambda}{}_{\lambda}\right). \tag{4.27}$$

Finally, we wish to express $t_{\mu\nu}$ in the weak field limit. Using (4.20) and (4.21) in (4.18) and (4.19), and keeping only the lowest order terms, we find

$$16\pi G\, t_{\mu\nu} = \frac{1}{2}\eta^{\lambda\rho}\eta^{\sigma\tau} h_{\lambda\sigma,\mu} h_{\rho\tau,\nu} - \frac{1}{4} A_{\mu} A_{\nu} - \eta_{\mu\nu} L_{g} \tag{4.28}$$

and,

$$L_{g} = \frac{1}{4}\eta^{\lambda\rho}\eta^{\sigma\tau} h_{\lambda\sigma,\alpha} h_{\rho\tau,}{}^{\alpha} - \frac{1}{8} A_{\alpha} A^{\alpha}\,. \tag{4.29}$$

Far Field Approximation

We now make the further approximation that the gravitational energy-momentum tensor $t_{\mu\nu}$ is to be evaluated far from the source $T_{\mu\nu}$. The procedure and notation are exactly as in the Brans-Dicke theory earlier. Equations (2.20) - (2.25) and (2.32) apply. In particular, we recall (2.25) and (2.32) [see Section 2 for details].

$$h_{\mu\nu,\sigma} = k_\sigma \dot{h}_{\mu\nu} \tag{4.30}$$

and

$$k_\rho k^\rho = 0\,, \tag{4.31}$$

which are valid far from the source.

Using (4.30), A_α in the far field is from (4.23)

$$A_\alpha = k_\alpha \dot{h}^\lambda{}_\lambda\,. \tag{4.32}$$

Therefore,

$$A_\alpha A^\alpha = k_\alpha k^\alpha \left(\dot{h}^\lambda{}_\lambda\right)^2 = 0 \tag{4.33}$$

from (4.31). Applying (4.30) and (4.31) to the first term of (4.29), we find

$$\frac{1}{4}\eta^{\lambda\rho}\eta^{\sigma\tau}k_\alpha k^\alpha \dot{h}_{\lambda\sigma}\dot{h}_{\rho\tau} = 0\,. \tag{4.34}$$

We conclude from (4.33) and (4.34) that in the far field

$$L_g = 0\,. \tag{4.35}$$

It is now a simple matter to apply (4.30) once again to (4.28), and to introduce (4.32), to find that

$$16\pi G\, t_{\mu\nu} = \frac{1}{2}k_\mu k_\nu \left[\eta^{\lambda\rho}\eta^{\sigma\tau}\dot{h}_{\lambda\sigma}\dot{h}_{\rho\tau} - \frac{1}{2}\left(\dot{h}^\lambda{}_\lambda\right)^2\right]$$

or,

$$t_{\mu\nu} = \frac{k_\mu k_\nu}{32\pi G}\left[\dot{h}^{\rho\tau}\dot{h}_{\rho\tau} - \frac{1}{2}\left(\dot{h}^\lambda{}_\lambda\right)^2\right] \tag{4.36}$$

in the far field.

Comparison with General Relativity

We are now in a position to compare the weak, far field predictions of the Rosen theory with those of general relativity. To do so, we utilize the comparable results already derived for the Brans~Dicke theory in the limit that the free parameter ω goes to infinity.

Starting with $t_{\mu\nu}$, we first write (2.43) in terms of the tensor field $h_{\mu\nu}$ by substituting (2.41) for ψ. This leads to

$$t_{\mu\nu} = \frac{k_\mu k_\nu}{16\pi}\varphi_0\left\{-\left(h^{\lambda\kappa}\ddot{h}_{\lambda\kappa} + \frac{1}{2}\dot{h}^{\lambda\kappa}\dot{h}_{\lambda\kappa}\right) + \right.$$
$$\left. + \frac{1+\omega}{(1+2\omega)^2}\left[2(1+\omega)h^{\lambda}{}_{\lambda}\ddot{h}^{\lambda}{}_{\lambda} + (2+\omega)\left(\dot{h}^{\lambda}{}_{\lambda}\right)^2\right]\right\}. \tag{4.37}$$

In the limit that ω goes to infinity, (4.37) goes over to

$$t_{\mu\nu}{}^{\mathrm{GR}} = -\frac{k_\mu k_\nu}{16\pi G}\left[h^{\lambda\kappa}\ddot{h}_{\lambda\kappa} + \frac{1}{2}\dot{h}^{\lambda\kappa}\dot{h}_{\lambda\kappa} - \frac{1}{2}h^{\lambda}{}_{\lambda}\ddot{h}^{\lambda}{}_{\lambda} - \frac{1}{4}\left(\dot{h}^{\lambda}{}_{\lambda}\right)^2\right], \tag{4.38}$$

where we have used also (2.16) for φ_0. Likewise, the field equation (2.14) goes over to

$$\Box^2 h_{\mu\nu}{}^{\mathrm{GR}} = -16\pi G\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T^{\lambda}{}_{\lambda}\right), \tag{4.39}$$

and the conservation law is (2.60)

$$T^{\mu\nu}{}_{,\nu}{}^{\mathrm{GR}} = 0\ . \tag{4.40}$$

We note on comparing (4.38), (4.39), and (4.40) with (4.36), (4.27), and (4.22) that the weak, far field conservation law and field equation in the Rosen theory are identical to those in general relativity, but that the gravitational energy-momentum tensors apparently differ.

In fact, (4.36) and (4.38) are genuinely inequivalent expressions. There is no elementary relationship within general relativity or the Rosen theory that can be used to establish their identity. Yet, they are entirely equivalent in their global consequences as we shall demonstrate below.

The root of the ambiguity lies in the principle of equivalence (of gravitational and inertial forces), which is satisfied by both general relativity and Rosen's theory. It is a consequence of the equivalence principle that it is not possible to say with certainty how much of a perceived force is inertial and how much is gravitational at a localized point of space-time. For example, it is always possible to define the coordinate system so that all forces at a specific point and time are reduced to zero. If the gravitational energy-momentum were described by a true tensor, the result at that point would have to be unambiguously zero in all frames. But, by assumption, it is not. Therefore the gravitational (pseudo) tensor is locally ambiguous and, in this case, the difference between (4.36) and (4.38) constitutes a specific example of that ambiguity.

Globally, the Rosen theory and general relativity make identical predictions as we now show.

We showed in Section 2 that, if the motion is periodic, it is appropriate to average $t_{\mu\nu}$. In this connection we showed that

$$\left\langle h^{\lambda\kappa}\ddot{h}_{\lambda\kappa}\right\rangle = -\left\langle \dot{h}^{\lambda\kappa}\dot{h}_{\lambda\kappa}\right\rangle . \tag{4.41}$$

(Equation (2.47)). By the same argument, it follows that

$$\left\langle h^{\lambda}{}_{\lambda}\ddot{h}^{\lambda}{}_{\lambda}\right\rangle = -\left\langle \left(\dot{h}^{\lambda}{}_{\lambda}\right)^2\right\rangle . \tag{4.42}$$

As a consequence of these, the average of (4.38) can be written

$$\left\langle t_{\mu\nu}{}^{\mathrm{GR}}\right\rangle = \frac{k_\mu k_\nu}{32\,\pi G}\left\langle \dot{h}^{\lambda\kappa}\dot{h}_{\lambda\kappa} - \frac{1}{2}\left(\dot{h}^{\lambda}{}_{\lambda}\right)^2\right\rangle . \tag{4.43}$$

Now let us assume that $h_{\mu\nu}$ has a continuous spectral distribution as given by (3.4) and (3.5). We argued in Section 3 that in such a case it is appropriate to compute the power spectral density, which involves integration of $t_{\mu\nu}$ over all time. In the Rosen theory we have from (4.36), and (3.12) and (3.14), that

$$\int_{-\infty}^{+\infty} dt' t_{\mu\nu} = -\frac{k_\mu k_\nu}{32\pi G} \int\int_{-\infty}^{+\infty} d\varpi\, d\varpi'\, \varpi\varpi' \Big[e^{\rho\tau}(\vec{x},\varpi) e_{\rho\tau}(\vec{x},\varpi') -$$

$$-\tfrac{1}{2} e^{\lambda}{}_{\lambda}(\vec{x},\varpi) e^{\lambda}{}_{\lambda}(\vec{x},\varpi') \Big] \int_{-\infty}^{+\infty} dt' \exp\big(i(\varpi+\varpi')t'\big) =$$

$$= \frac{k_\mu k_\nu}{16\ G} \int_{-\infty}^{+\infty} d\varpi\, \varpi^2 \Big[e^{\rho\tau}(\vec{x},\varpi) e^{*}_{\rho\tau}(\vec{x},\varpi) - \tfrac{1}{2} \left| e^{\lambda}{}_{\lambda}(\vec{x},\varpi) \right|^2 \Big]_{\text{Rosen}} \quad , \qquad (4.44)$$

where $t' = k_\lambda x^\lambda$. In general relativity, the same operations on (4.38) give

$$\int_{-\infty}^{+\infty} dt' t_{\mu\nu}{}^{\text{GR}} = \frac{k_\mu k_\nu}{16\pi G} \int\int_{-\infty}^{+\infty} d\varpi\, d\varpi' \Big[(\varpi')^2 + \tfrac{1}{2}\varpi\varpi' \Big] \times$$

$$\times \Big[e^{\lambda\kappa}(\vec{x},\varpi) e_{\lambda\kappa}(\vec{x},\varpi') - \tfrac{1}{2} e^{\lambda}{}_{\lambda}(\vec{x},\varpi) e^{\lambda}{}_{\lambda}(\vec{x},\varpi') \Big] \int_{-\infty}^{+\infty} dt' \exp\big(i(\varpi+\varpi')t'\big) =$$

$$= \frac{k_\mu k_\nu}{16\, G} \int_{-\infty}^{+\infty} d\varpi\, \varpi^2 \Big[e^{\lambda\kappa}(\vec{x},\varpi) e^{*}_{\lambda\kappa}(\vec{x},\varpi) - \tfrac{1}{2} \left| e^{\lambda}{}_{\lambda}(\vec{x},\varpi) \right|^2 \Big]_{\text{GR}} \quad . \qquad (4.45)$$

We see on comparing (4.43) with (4.36) and (4.45) with (4.44) that, though $t_{\mu\nu}$ has not been defined identically in the two theories, there are no differences whatever in the global predictions for periodic and aperiodic systems given the field. Moreover, the equations that govern the field ((4.27) and (4.39)) and conservation ((4.22) and (4.40)) are identical also, so that in the weak, far field approximation the bi-metric theory of Rosen makes predictions identical to those of general relativity.

Applications

Therefore, specific relationships for quadrupole radiation in the Rosen theory follow by letting ω go to infinity in our previous results for the Brans~Dicke theory. For periodic systems we have from (2.84) that

$$\left\langle \frac{dE}{dt} \right\rangle_{\text{Rosen}} = \frac{G}{15} \left\langle 3\left(\dddot{Q}^{ij}\right)^2 - \left(\dddot{Q}^{k}{}_{k}\right)^2 \right\rangle. \qquad (4.46)$$

(Symbols defined in Section 2). For the example of a binary system, we have from (2.126)

$$\left\langle \frac{dE}{dt} \right\rangle_{\text{Rosen}} = \frac{32}{5} \frac{G^4 m^2 M^7}{a^5 (m+M)^4} \left[\frac{1 + \frac{73}{24}\varepsilon^2 + \frac{37}{96}\varepsilon^4}{\left(1-\varepsilon^2\right)^{7/2}} \right]. \qquad (4.47)$$

This can be related to the change of orbital period $\dot{T}$ by using (2.130). In the case of PSR 1913+16, (2.134) gives

$$\left\langle \frac{dE}{dt} \right\rangle_{\text{Rosen}} = 1.53 \times 10^{32} \ \frac{\text{g cm}^2}{\text{sec}^3}, \qquad (4.48)$$

which leads to

$$\dot{T} = 3.36 \times 10^{-12} \ \text{sec/sec} \qquad \text{(Rosen)} \qquad (4.49)$$

from (2.136).

Finally, the 2-sided power spectral density function associated with a system of colliding particles follows from (3.61) by letting ω go to infinity. The result, which is the same as for general relativity, is

$$P(\varpi)^{\text{Rosen}} = \frac{G}{4\pi} \sum_{N,M} \eta_N \eta_M m_N m_M \frac{1+\beta_{NM}^2}{\left(1-\beta_{NM}^2\right)^{1/2} \beta_{NM}} \ln\left(\frac{1-\beta_{NM}}{1+\beta_{NM}} \right). \qquad (4.50)$$

(Symbols defined in Section 3).

5.0 **Conclusion**

Dipole gravitational radiation from a binary system in both the Brans-Dicke and Rosen theories is proportional to the square of the difference of the self-gravitational binding energy per unit mass for the two bodies (9). That is, the dipole radiation is proportional to

$$\left(\frac{\Omega_1}{m_1} - \frac{\Omega_2}{m_2}\right)^2 , \tag{5.1}$$

where m_1 and m_2 are the masses of the bodies,

$$\Omega = \frac{1}{2} \int_{\text{body}} d^3\vec{x}\, d^3\vec{x}' \frac{\rho(\vec{x})\rho(\vec{x}')}{|\vec{x}' - \vec{x}|} , \tag{5.2}$$

and $\rho(\vec{x})$ is the distribution of rest-mass density within the body. Moreover, this radiation is proportional also to the square of the first ((11), Brans-Dicke) or second ((12), Rosen) derivative of the relative separation vector. It follows that there are two circumstances under which there might be coincidentally no detectable dipole radiation from a binary system, though the Brans-Dicke or Rosen theory might apply: circularity of the orbits or a fortuitous combination of mass and mass distribution within each body such that (5.1) is negligible. In the case of PSR 1913+16, the first possibility is obviated by the known high eccentricity, ε = 0.617155, (1). However, until the precise internal structures of both bodies of the pair are known, the second possibility remains open. We cannot at this time make a definitive choice between the Brans-Dicke theory, the Rosen theory, and general relativity on the basis of the prediction of dipole radiation in the first two. This situation might change in the future if a sufficient number of systems of this type are discovered that either there is clear evidence of the dipolar contribution, or a strong statistical argument can be advanced against multiple instances of the two fortuitous circumstances in which dipole radiation does not occur.

As our results show, radiation of the quadrupole order in all three theories – that is, Brans-Dicke, Rosen, and general relativity – occurs independently of the detailed internal structure of the participating bodies. Consequently, knowledge of the theoretical quadrupole radiation rate in each theory might enable, on the basis of observation of the one binary system PSR 1913+16, an unambiguous determination of which theory is obeyed by nature.

Unfortunately for this program, the Rosen and general relativistic predictions are identical. No matter how good the observational data, we shall never be able to distinguish between these two theories on the basis of quadrupole radiation.

In the Brans-Dicke theory on the other hand, the radiation rate is a function of the parameter ω. As can be seen from Figure l or (2.135), the predicted range of the derivative of the orbital period of PSR 1913+16 as a function of positive ω due to quadrupole radiation is

$$3.36 \times 10^{-12} - 2.76 \times 10^{-12} = 0.6 \times 10^{-12} \ \mathrm{sec/sec},$$

whereas the observational uncertainty established by Taylor *et al.* (1) encompasses a range

$$3.8 \times 10^{-12} - 2.6 \times 10^{-12} = 1.2 \times 10^{-12} \ \mathrm{sec/sec}.$$

The observational uncertainty currently exceeds the range attainable through adjustment of ω by 2:1. Once again, we cannot make a definitive statement regarding the Brans-Dicke theory on the basis of current observations of PSR 1913+16 and the assumption that the dominant radiation mode is quadrupolar.

It has been argued recently on the basis of independent observations (18) that ω must exceed a value of about 300. If this is so, the Brans-Dicke and general relativistic predictions for PSR 1913+16 become very nearly equal, as can be seen from Figure 1, and the chances of ever being able to distinguish between these two theories in this manner appear remote indeed.

Finally, we have developed the power spectral density for a system of colliding particles in the Brans-Dicke and Rosen theories, again based on the weak field approximation. The result in the Rosen theory is identical to that in general relativity. Indeed we have shown that this is the case for all predictions of the Rosen theory in the weak, far field approximation.

The result in the Brans-Dicke theory, (3.61), exhibits dependency on the free parameter ω. For comparison with general relativity, we can write (3.61) as

$$P(\varpi) = \frac{G}{4\pi} \sum_{N,M} \eta_N \eta_M m_N m_M \frac{A + B\beta_{NM}^2}{\left(1-\beta_{NM}^2\right)^{1/2} \beta_{NM}} \ln\left(\frac{1-\beta_{NM}}{1+\beta_{NM}}\right) \tag{5.3}$$

where:

$$A = \frac{(2\omega + 7)(\omega + 1)}{(2 + \omega)(3 + 2\omega)} \quad \text{and} \quad B = \frac{2\omega^2 + 3\omega + 2}{(2 + \omega)(3 + 2\omega)} . \tag{5.4}$$

Figure 2 is a plot of coefficients A and B as functions of positive ω. We see that for ω greater than 300, A and B are very close to 1, their values in the general relativistic limit (4.50). We conclude that if astronomical objects are ever observed to radiate through collision processes, the present bound on ω will again preclude discrimination between the Brans-Dicke and general relativistic theories in this order of approximation.

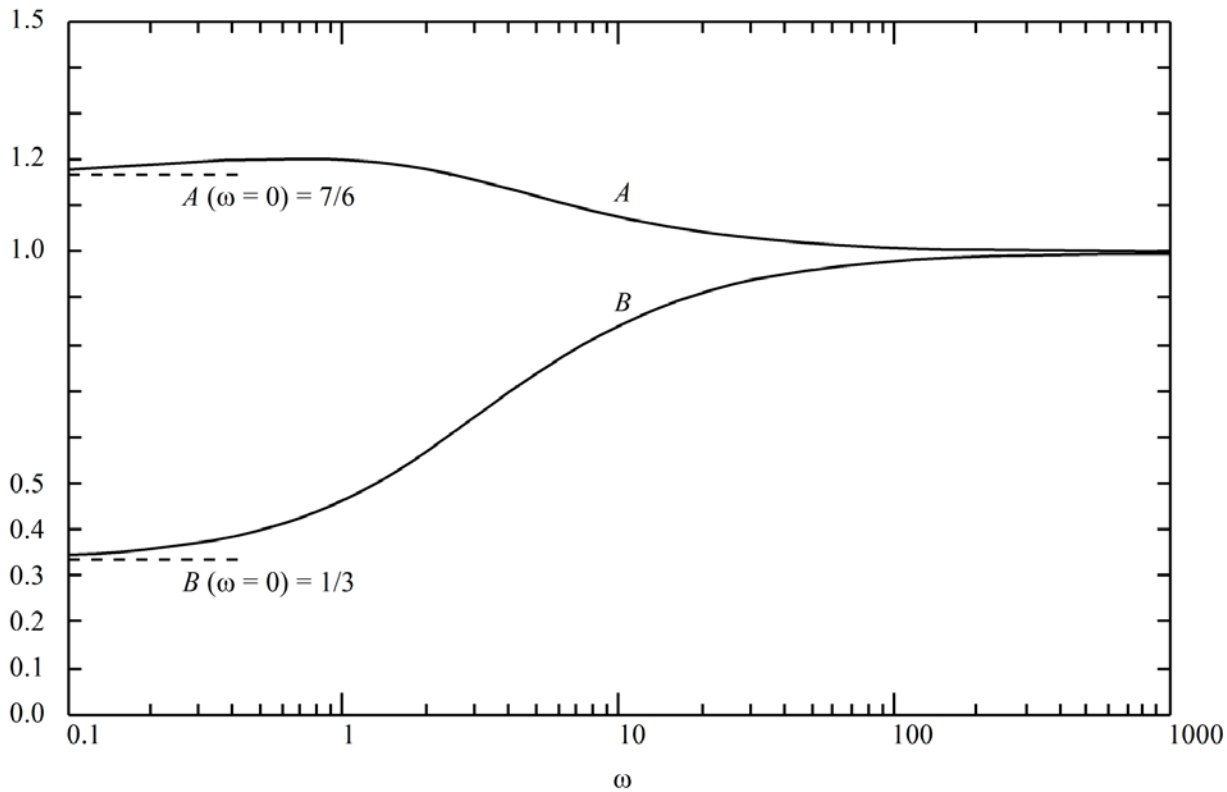

Figure 2. Coefficients *A* & *B*, collision problem, Brans-Dicke theory.

APPENDICES

A. Consequences of the Conservation Law $T^{\mu\nu}{}_{,\nu} = 0$.

B. Evaluation of an Integral Relating to a Binary System.

C. Evaluation of an Integral Relating to a System of Colliding Particles.

Appendix A

Consequences of the Conservation Law:

$$T^{\mu\nu}{}_{,\nu} = 0\ . \tag{A.1}$$

(A.1) may be written

$$T^{\mu 0}{}_{,0} + T^{\mu i}{}_{,i} = 0\ , \tag{A.2}$$

which we can specialize to

$$T^{00}{}_{,0} + T^{0i}{}_{,i} = 0 \tag{A.3}$$

and

$$T^{j0}{}_{,0} + T^{ji}{}_{,i} = 0\ . \tag{A.4}$$

Multiply (A.4) by x^k, integrate over all space, and discard the surface terms on the assumption that $T^{\mu\nu}$ goes to zero sufficiently rapidly at spatial infinity. We find

$$\int d^3\vec{x}\, x^k T^{j0}{}_{,0} = \frac{\partial}{\partial t}\int d^3\vec{x}\, x^k T^{j0} = -\int d^3\vec{x}\, x^k T^{ji}{}_{,i} =$$
$$= -\left(x^k T^{ji} - \int d^3\vec{x}\, x^k{}_{,i} T^{ji}\right)$$

or,

$$\boxed{\int d^3\vec{x}\, T^{jk} = \frac{\partial}{\partial t}\int d^3\vec{x}\, x^k T^{j0}} \tag{A.5}$$

where we have used also

$$x^k{}_{,i} = \delta^k{}_i\ . \tag{A.6}$$

Now multiply (A.3) by $x^j x^k$ and likewise integrate to get

$$\int d^3\vec{x}\, x^j x^k T^{00}{}_{,0} = \frac{\partial}{\partial t}\int d^3\vec{x}\, x^j x^k T^{00} = -\int d^3\vec{x}\, x^j x^k T^{0i}{}_{,i} =$$

$$= -\left[x^j x^k T^{0i} - \int d^3\vec{x} \left(x^j{}_{,i} x^k + x^j x^k{}_{,i} \right) T^{0i} \right]$$

or,

$$\int d^3\vec{x} \left(x^k T^{0j} + x^j T^{0k} \right) = \frac{\partial}{\partial t} \int d^3\vec{x}\, x^j x^k T^{00} \,. \tag{A.7}$$

Since T^{jk} is symmetric in jk, we may write (A.5) as

$$\int d^3\vec{x}\, T^{jk} = \frac{1}{2} \frac{\partial}{\partial t} \int d^3\vec{x} \left(x^k T^{0j} + x^j T^{0k} \right). \tag{A.8}$$

Substitution of (A.7) into the right side of (A.8) gives

$$\boxed{\int d^3\vec{x}\, T^{jk} = \frac{1}{2} \frac{\partial^2}{\partial t^2} \int d^3\vec{x}\, x^j x^k T^{00}} \tag{A.9}$$

If we multiply (A.3) by x^k and integrate we find

$$\int d^3\vec{x}\, x^k T^{00}{}_{,0} = \frac{\partial}{\partial t} \int d^3\vec{x}\, x^k T^{00} = -\int d^3\vec{x}\, x^k T^{0i}{}_{,i} =$$

$$= -\left(x^k T^{0i} - \int d^3\vec{x}\, x^k{}_{,i} T^{0i} \right)$$

or,

$$\boxed{\int d^3\vec{x}\, T^{0k} = \frac{\partial}{\partial t} \int d^3\vec{x}\, x^k T^{00}} \tag{A.10}$$

We have defined in the text the moments of T^{00}

$$M(t) \equiv \int d^3\vec{x}\, T^{00}(\vec{x}, t) \tag{A.l1}$$

$$D^k(t) \equiv \int d^3\vec{x}\, x^k T^{00}(\vec{x}, t) \tag{A.12}$$

$$Q^{ij}(t) \equiv \int d^3\vec{x}\, x^i x^j T^{00}(\vec{x},t). \qquad \text{(A.13)}$$

In terms of these definitions, we have from (A.9) and (A.10) that

$$\boxed{\int d^3\vec{x}\, T^{jk} = \tfrac{1}{2}\ddot{Q}^{ij}(t)} \qquad \text{(A.14)}$$

and

$$\boxed{\int d^3\vec{x}\, T^{0k} = \dot{D}^k(t)} \qquad \text{(A.15)}$$

where dot denotes $\partial/\partial t$.

Finally, we consider $\dot{M}$ and $\ddot{D}^k$. From (A.l1) and (A.3), we find

$$\dot{M}(t) = \int d^3\vec{x}\, T^{00}{}_{,0} = -\int d^3\vec{x}\, T^{0i}{}_{,i} \;. \qquad \text{(A.16)}$$

(A.16) can be expressed as a surface integral using Gauss' Theorem. By letting the surface approach infinity and assuming that $T^{\mu\nu}$ goes to zero sufficiently rapidly at infinity, we conclude that

$$\boxed{\dot{M}(t) = 0} \qquad \text{(A.17)}$$

Likewise, take the time derivative of (A.15) and use (A.4) to get

$$\ddot{D}^k(t) = \int d^3\vec{x}\, T^{0k}{}_{,0} = -\int d^3\vec{x}\, T^{ik}{}_{,i} \;. \qquad \text{(A.18)}$$

By the argument just used with (A.16), we conclude from (A.18) that

$$\boxed{\ddot{D}^k(t) = 0} \qquad \text{(A.19)}$$

Appendix B

Evaluation of an Integral Relating to a Binary System:

$$I = \int_0^{\pi} dE \, \frac{\sin^2 E}{\left(1 - \varepsilon \cos E\right)^5} . \tag{B.1}$$

First, consider

$$\frac{d}{dE}\left(1 - \varepsilon \cos E\right)^{-n} = -n\left(1 - \varepsilon \cos E\right)^{-(n+1)}\left(\varepsilon \sin E\right)$$

or,

$$\left(1 - \varepsilon \cos E\right)^{-(n+1)} = -\frac{1}{n\,\varepsilon \sin E}\frac{d}{dE}\left(1 - \varepsilon \cos E\right)^{-n} . \tag{B.2}$$

Set $n = 4$ in (B.2), substitute into (B.1), and do one integration by parts to get

$$I = \frac{1}{4\varepsilon}\int_0^{\pi} \frac{\cos E}{\left(1 - \varepsilon \cos E\right)^4} dE \; . \tag{B.3}$$

Now introduce the change of variable

$$x = 1 - \varepsilon \cos E$$

into (B.3) to get

$$I = \frac{1}{4\pi^2}\left(\int_{1-\varepsilon}^{1+\varepsilon} \frac{dx}{x^4\sqrt{R}} - \int_{1-\varepsilon}^{1+\varepsilon} \frac{dx}{x^3\sqrt{R}}\right), \tag{B.4}$$

where

$$R = \left(\varepsilon^2 - 1\right) + 2x - x^2 .$$

The two integrals in (B.4) can be reduced to integrals of the form

$$\int_{1-\varepsilon}^{1+\varepsilon} \frac{dx}{x\sqrt{R}} \tag{B.5}$$

by using entry 231, 9a of (17) recursively. (B.5) is evaluated using entry 231, l0a of (17). One finds, finally, that

$$I = \frac{\pi}{8} \frac{4 + \varepsilon^2}{\left(1 - \varepsilon^2\right)^{7/2}} . \qquad \text{(B.6)}$$

Appendix C

Evaluation of an Integral Relating to a System of Colliding Particles:

$$I = \int d\Omega \frac{\left(P_N^{\lambda} P_{M\lambda}\right)^2}{\left(P_N^{\lambda} k_{\lambda}\right)\left(P_M^{\lambda} k_{\lambda}\right)} \tag{C.1}$$

where

$$k_0 = 1, \qquad \vec{k} = -\hat{x}, \qquad \hat{x}\cdot\hat{x} = 1 \tag{C.2}$$

$$\left.\begin{array}{ll} P_N^0 = m_N \gamma_N, & \vec{P}_N = m_N \gamma_N \vec{v}_N \\ P_M^0 = m_M \gamma_M, & \vec{P}_M = m_M \gamma_M \vec{v}_M \end{array}\right\} \tag{C.3}$$

$$\gamma \equiv \left(1 - v^2\right)^{-1/2}. \tag{C.4}$$

The integral I can be written

$$I = m_N\, m_M\, \gamma_N\, \gamma_M \left(1 - \vec{v}_N \cdot \vec{v}_M\right)^2 I_1 \tag{C.5}$$

where

$$I_1 = \int d\Omega \frac{1}{\left(1 - \vec{v}_N \cdot \hat{x}\right)\left(1 - \vec{v}_M \cdot \hat{x}\right)}. \tag{C.6}$$

Pick the coordinate system so that $\vec{v}_N$ is along the 3-axis and $\vec{v}_M$ is in the 2,3-plane. To ease the notation a bit, re-label $\vec{v}_N$ and $\vec{v}_M$ as $\vec{a}$ and $\vec{b}$, respectively, so that

$$\left.\begin{array}{l} \vec{v}_N = \vec{a} = a_3\, \hat{e}_3 \\ \vec{v}_M = \vec{b} = b_2\, \hat{e}_2 + b_3\, \hat{e}_3 \end{array}\right\} \tag{C.7}$$

$$\left.\begin{array}{c} \hat{x} = \sin\theta\, \cos\varphi\, \hat{e}_1 + \sin\theta\, \sin\varphi\, \hat{e}_2 + \cos\theta\, \hat{e}_3 \\ d\Omega = \sin\theta\; d\theta\, d\varphi \end{array}\right\}. \tag{C.8}$$

I_1 may now be written in terms of (C.7) and (C.8) as

$$I_1 = \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} \frac{d\varphi}{\alpha - \beta \sin\varphi} \tag{C.9}$$

where:

$$\left.\begin{aligned} \alpha &= \left(1 - a_3 \cos\theta\right)\left(1 - b_3 \cos\theta\right) \\ \beta &= \left(1 - a_3 \cos\theta\right) b_2 \sin\theta \end{aligned}\right\}. \tag{C.10}$$

The integral on φ is[8]

$$\int_0^{2\pi} \frac{d\varphi}{\alpha - \beta \sin\varphi} = \frac{2\pi}{\sqrt{\alpha^2 - \beta^2}} \tag{C.11}$$

so that I_1 is

$$I_1 = 2\pi \int_0^{\pi} \sin\theta \, d\theta \frac{1}{\left(1 - a_3 \cos\theta\right)\sqrt{\left(1 - b_3 \cos\theta\right)^2 - b_2^2\left(1 - \cos^2\theta\right)}}. \tag{C.12}$$

This may be written

$$I_1 = 2\pi \int_{\left(1+a_3\right)^{-1}}^{\left(1-a_3\right)^{-1}} \frac{dx}{\sqrt{a + bx + cx^2}} \tag{C.13}$$

by using the change of variable

$$x = \frac{1}{1 - a_3 \cos\theta}. \tag{C.14}$$

One finds that

$$\left.\begin{aligned} a &= b_2^2 + b_3^2 \\ b &= 2\left[a_3 b_3 - \left(b_2^2 + b_3^2\right)\right] \\ c &= \left(a_3 - b_3\right)^2 + b_2^2\left(1 - a_3^2\right) \end{aligned}\right\} \tag{C.15}$$

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[8] Ref. (17), entry 331, 41a, *Bestimmte Integrale*.

Finally, (C.13) may be evaluated with the aid of[9]

$$\int \frac{dx}{\sqrt{R}} = \frac{1}{\sqrt{c}} \ln\left(2\sqrt{cR} + 2cx + b\right) \tag{C.16}$$

where:

$$R = a + bx + cx^2 .$$

By utilizing the properties of the 4-vectors (C.3) along with (C.5), and by defining the relative speed

$$\beta_{NM} \equiv \left[1 - \frac{m_N^2 \, m_M^2}{\left(P_N^{\lambda} \, P_{M\lambda}\right)^2}\right]^{1/2} , \tag{C.18}$$

one can show that

$$I = \int d\Omega \, \frac{\left(P_N^{\lambda} \, P_{M\lambda}\right)^2}{\left(P_N^{\lambda} \, k_{\lambda}\right)\left(P_M^{\lambda} \, k_{\lambda}\right)} = \frac{2\pi \, m_N \, m_M}{\left(1 - \beta_{NM}^2\right)^{1/2} \beta_{NM}} \ln\left(\frac{1 - \beta_{NM}}{1 + \beta_{NM}}\right). \tag{C.19}$$

[9] Ref. (16), entry 2.261.

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